Assignment 1: due Thursday February 12

1. Write up a complete proof of term-by-term differentiability, starting from: Let $\epsilon > 0$. We must find $\delta > 0$ such that $|w - z| < \delta$ implies

$$\left| \frac{f(w) - f(z)}{w - z} - g(z) \right| < \epsilon.$$

Here, $f(z) = \sum_0^\infty c_n z^n$ has radius of convergence $R > 0$, $|z| < R$, $|w| < R$, and $g(z)$ is given by the derived series.

2. *Definition of* $\partial_{\overline{z}}$ *(and* ∂_z*).* Among all expressions (vector fields in the plane with $\mathbb{C}$-valued coefficients) $L = \phi(x, y)\partial_x + \psi(x, y)\partial_y$, there is exactly one with

a) $L(f) = 0$ wherever f is complex-differentiable,

b) $L(\overline{z}) = 1$.

This L is $\partial_{\overline{z}}$. Determine the coefficients $\phi(x, y)$ and $\psi(x, y)$.

3. Assume that $f(z) = \sum_0^\infty c_n z^n$ and $g(z) = \sum_1^\infty a_n z^n$ (so $g(0) = 0$) have positive radii of convergence R and r respectively.

a) Determine the power series for $f \circ g$ in powers of z.

b) Let $g(z) = \alpha z$ ($\alpha \in \mathbb{C}$). What is the radius of convergence of the series for $f \circ g$?

c) Answer the question in b) for general g.

4. a) Show that for a sequence of real numbers q_n,

$$\limsup_{n \to \infty} q_n = \lambda$$

if and only if the following pair of statements hold:

i) For every $\epsilon > 0$, the set of n for which $q_n \geq \lambda + \epsilon$ is a finite set.

ii) There is a subsequence $\{q_{n(k)}\}$ of $\{q_n\}$ that converges to λ.

b) Use this characterization of lim sup to derive the formula for the radius of convergence of a power series:

$$R^{-1} = \limsup_{n \to \infty} |c_n|^{1/n}.$$

5. Suppose that $\sum_{n=0}^\infty c_n z^n$ has radius of convergence $R > 0$.

a) If $|z_0| = r < R$, show that the double summation

$$\sum_{n=0}^{\infty} c_n \sum_{m=0}^{n} \binom{n}{m} z_0^{n-m} (z - z_0)^m$$

converges absolutely whenever $|z - z_0| < R - r$.

b) Give a counterexample to the assertion: *The radius of convergence of*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=m}^{\infty} c_n \binom{n}{m} z_0^{n-m} \right\} (z - z_0)^m$$

is $R - r$.