Assignment 8 (due April 13)

I wish to point out that that the change of variables formula involves two limits. First, the definition of the Riemann integral involves the approximation of continuous functions by constant functions on small rectangles (in the case of integrals with respect to area), or on small cubes (in the case of integrals with respect to volume), or ... never mind in higher dimensions.

The change of variables requires a further approximation of a differentiable transformation $T$ defined on a domain in $\mathbb{R}^n$ by its linearization (tangent plane approximation) on small regions, under which $n$-dimensional volume is “distorted” by the absolute value of the determinant. The function $\det T$ is called the Jacobian of $T$. (See the pictures in Section 5.5.)

Do the following problems:

1. Let $W$ be the solid tetrahedron of Example 4 (page 318). Let $D$ be the solid cube whose eight vertices are at the points with coordinates $(a, b, c)$ satisfying $a, b, c \in \{0, 1\}$.
   (a) Calculate the volume ratio $Vol(W)/Vol(D)$ via the use of iterated integrals.
   (b) Interpret the outcome of (a) geometrically.

2. Let $W$ be the solid from Example 6 (page 320).
   (a) Show that the slices of $W$ by planes parallel to the $(x, y)$-plane are discs, points, or empty.
   (b) Convert the volume integral to cylindrical coordinate and check that the outcome of the calculation of this integral is the same as the one in Example 6.

3. Let $D_r$ denote the intersection of the disc of radius $r$ centered at the origin with the closed first quadrant $Q = [0, \infty) \times [0, \infty) \subset \mathbb{R}^2$, and $C_r$ be the square $[0, r] \times [0, r]$.
   (a) Show that $D_r \subset C_r$ and that $C_r \subset D_{2r}$.
   (b) Let $f \geq 0$ be a continuous non-negative function on the closed first quadrant. Show that (if we allow $\infty$ as a possibility for the limits),

   \[
   \lim_{r \to \infty} \iint_{D_r} f \, dA = \lim_{r \to \infty} \iint_{C_r} f \, dA.
   \]

   (c) Do problems #30 and #33 in Section 5.8 from Colley.

4. Do the following additional problems from Colley:
   Sec. 5.4 18, 22, 25(a),(c),(d)
   Sec. 5.5 6, 8, 22
   Sec. 5.6 4, 22