Riemann and Green

First, let’s understand that integration with respect to area has been defined rigorously. That is all I will say about Riemann here. Be sure to do #1.

0. If you are feeling ambitious, search for a proof of the fact that a piecewise $C^1$ curve has length zero, in the sense of Figure 5.37 (page 306), and write it up in your own words. [You may need to use the offciial definition of compactness in terms of coverings; look it up in a source of higher level than a Calculus text.]

1. Let $\vec{F}$ be a continuous vector field defined on (say) a simple $C^1$ path $\vec{x}: I \to \mathbb{R}^2$.

   (a) Show that
   \[
   \left| \int_{\vec{x}} \vec{F} \cdot d\vec{s} \right| \leq \max_{t \in I} \{ \| \vec{F}(\vec{x}(t)) \| \} \cdot L(\vec{x}),
   \]
   where $L$ denotes the length of the path.

   (b) Let $\epsilon > 0$ be given. Show that there is a piecewise linear path $\vec{p}: I \to \mathbb{R}^2$ such that $\| \vec{p}(t) - \vec{x}(t) \| < \epsilon$, and also $\| \vec{p}'(t) - \vec{x}'(t) \| < \epsilon$ (except, of course, at the corners in $\vec{p}$). [Hint: Section 3.2]

2. Let’s reduce the proof of Green’s theorem to something close to the special case of rectangles. So let $D$ be a domain in $\mathbb{R}^2$ with piecewise $C^1$ boundary $\partial D$. In the statement of Green’s theorem, we’ll make the mild assumption that $M$ and $N$ are $C^1$ functions on a larger set than $D$, namely on an open set $U_{\epsilon}(D)$ of points in $\mathbb{R}^2$ that lie within some fixed distance $r > 0$ from $D$ (so $D \subset U_{\epsilon}(D)$).

   (a) We must specify the orientation of the boundary first (see p.381), so if we then use a parametrization of $\partial D$, it must be compatible with the orientation. Give a definition of the orientation that does not use the picture. What does “to the left” mean intrinsically? (It may help to think in terms of normal vectors.)

   (b) Let $P$ be a polygonal region in $U_{\epsilon}(D)$. Show that Green’s theorem holds for $P$.

   (c) Show that for general $D$, given $\epsilon' > 0$, there is a polygonal region $P \subset U_{\epsilon}(D)^2$ for which both
   \[
   \left\| \int_{\partial P} \vec{F} \cdot d\vec{s} - \int_{\partial D} \vec{F} \cdot d\vec{s} \right\| < \epsilon',
   \]
   and the analogous inequality for the area integrals over $P$ and $D$ hold. (This is similar to what is mentioned in Colley, on page 388.) Then deduce from part (b) that for that choice of $P$, we get
   \[
   \left| \int_D - \int_{\partial D} \right| < 2\epsilon'.
   \]
   But $\epsilon'$ was arbitrary; let $\epsilon' \to 0$.

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1By the way, the image of a closed set under a continuous mapping need not be closed; the image of a bounded set need not be bounded. But in the settings we consider in our course, the image of a set that is both closed and bounded is closed and bounded.

2$P$ need not be contained in $D$; it’s an issue of convexity.