That important point

You may recall that I got stuck over an important detail in Wednesday’s class. The point is not really very difficult at all; I just hadn’t thought out the details in advance. The goal was to prove:

**Proposition.** Let \( f : X \to \mathbb{R} \), where \( X \) is a neighborhood of \((x_0, y_0)\) in \( \mathbb{R}^2 \) be a function for which the gradient vector \( \nabla f(x_0, y_0) \) exists (i.e., is defined). A necessary condition for \( f \) to be differentiable at \((x_0, y_0)\) is that for every unit vector \( \vec{u} \in \mathbb{R}^2 \), the directional derivative \( D_{\vec{u}} f(x_0, y_0) \) exists and equals \( \nabla f(x_0, y_0) \cdot \vec{u} \).

**Proof.** We are given that \( D_{\vec{u}} f(x_0, y_0) \) exists for \( \vec{u} = e_1 \) and \( \vec{u} = e_2 \) exist at \((x_0, y_0)\); these are just the partial derivatives of \( f \) at \((x_0, y_0)\). We checked that two vectors parallel to the would-be tangent plane are

\[
(*) \quad \vec{v}(\vec{e}_1) = (1, 0, f_x(x_0, y_0)) \quad \text{and} \quad \vec{v}(\vec{e}_2) = (0, 1, f_y(x_0, y_0)).
\]

If we regard \( \mathbb{R}^3 \) as \( \mathbb{R}^2 \times \mathbb{R} \) (indeed, the graph of an arbitrary function \( g : S \to T \) is a subset of \( S \times T \), viz.

\[
\{(s, t) \in S \times T \mid t = g(s)\},
\]

which projects one-to-one and onto \( S \). We can then write

\[
\vec{v}(\vec{e}_1) = (\vec{e}_1; f_x(x_0, y_0)) \quad \text{and} \quad \vec{v}(\vec{e}_2) = (\vec{e}_2; f_y(x_0, y_0)).
\]

We invoke the anti-discrimination clause to see that the vector in \( \mathbb{R}^3 \) that is parallel to the would-be tangent plane and projects onto \( \vec{u} \in \mathbb{R}^2 \) is

\[
\vec{v}(\vec{u}) = (\vec{u}; D_{\vec{u}}(x_0, y_0)).
\]

This can be in that plane only if \( \vec{v}(\vec{u}) \) is a linear combination of \( \vec{v}(\vec{e}_1) \) and \( \vec{v}(\vec{e}_2) \):

\[
\vec{v}(\vec{u}) = c_1 \vec{v}(\vec{e}_1) + c_2 \vec{v}(\vec{e}_2),
\]

where \( c_1, c_2 \in \mathbb{R} \). Breaking the above into components gives \( c_1 = u_1 \) and \( c_2 = u_2 \). Using \((*)\), we get

\[
D_{\vec{u}}(x_0, y_0)) = u_1 f_x(x_0, y_0) + u_2 f_y(x_0, y_0),
\]

and the right-hand side is just \( \nabla f(x_0, y_0) \cdot \vec{u} \). \( \Box \)