Differentiability, especially in two variables

We have in Theorem 3.9 on page 127 the simplest criterion for differentiability. In the representative case of a function of two variables, it says: *Suppose that the partial derivatives f_x and f_y are continuous at the point (a, b). Then f is differentiable at (a, b).*

Here, we remind the reader that the definition of differentiability at (a, b) can be written as:

$$\frac{f(x,y) - f(a,b) - f_x(a,b)\Delta x - f_y(a,b)\Delta y}{\|(x,y) - (a,b)\|} \to 0$$

as $(x, y) \to (a, b)$. Here, I'm using the excess notation Δx for $x - a$ and Δy for $y - b$ in the hope that it helps. For the limit above, it can be stated *as Δx and Δy go to zero.*

The way one gets from the rather simple conditions for the criterion to the definition of derivative is by applying the *one-variable* Mean Value Theorem a couple of times, as was done in class on Wednesday. That is, we are giving a proof of the criterion, thereby justifying its use. (What a relief it is to have such a simple criterion!)

Let's finish the argument. Recall we showed by the MVT that for each (x, y) there are, somewhere out there, a number ξ between x and a, and an η between y and b, for which

$$f(x,y) - f(a,b) - f_x(\xi,b)\Delta x - f_y(a,\eta)\Delta y = 0.$$

(That's equals, not "is approaching", zero.) We use this to write:

$$\begin{aligned} &f(x,y) - f(a,b) - f_x(a,b)\Delta x - f_y(a,b)\Delta y \\ &\qquad = f_x(\xi,b)\Delta x + f_y(a,\eta)\Delta y - f_x(a,b)\Delta x - f_y(a,b)\Delta y \\ &\qquad\qquad = [f_x(\xi,b) - f_x(a,b)]\Delta x + [f_y(a,\eta) - f_y(a,b)]\Delta y. \end{aligned}$$

Note that as $x \to a$, $\xi \to a$ too. Similarly, $y \to b \Rightarrow \eta \to b$. We invoke the triangle inequality to shed excess detail:

$$\begin{aligned} &\frac{|f(x,y) - f(a,b) - f_x(a,b)\Delta x - f_y(a,b)\Delta y|}{\|(x,y) - (a,b)\|} \\ &\qquad = \frac{|[f_x(\xi,b) - f_x(a,b)]\Delta x + [f_y(a,\eta) - f_y(a,b)]\Delta y|}{\|(x,y) - (a,b)\|} \\ &\quad \le |f_x(\xi,b) - f_x(a,b)|\frac{|\Delta x|}{\|(x,y) - (a,b)\|} + |f_y(a,\eta) - f_y(a,b)|\frac{|\Delta y|}{\|(x,y) - (a,b)\|}. \end{aligned}$$

Use the continuity assumption and triangle to see this does go to zero (use the valid maneuver: if $u \to 0$ and v stays bounded, then $uv \to 0$). The definition of differentiability at (a, b) is satisfied.