**Constrained max/min**

We should all recall the proposition:

**Proposition.** *Let $D$ be a compact region in $\mathbb{R}^n$, with interior $U$ and $f : D \to \mathbb{R}$ a function that is $C^1$ in $U$. Then the maximum and minimum values of $f$ on $D$ exist. Each occurs at either a critical point for $f$ (where $\nabla f = \mathbf{0}$) in $U$ or a point of $\partial D$.*

The result for constrained max/min is somewhat similar. Unfortunately, we must bring in the Implicit Function Theorem to explain it properly. Let $Y \subset \mathbb{R}^n$ be the intersection of level sets of $m$ $C^1$ functions $g_1 = g_2 = \ldots g_m = 0$ (constraint conditions). Let $f$ be a function that is $C^1$ in some neighborhood of $Y$. If these $g_i$'s satisfy *a rank condition* at $\mathbf{x} \in Y$, then in a neighborhood of $x$ in $\mathbb{R}^n$, $Y$ is equal to the graph of $m$ coordinates of $\mathbb{R}^n$ as a $C^1$ function of the other $n - m$. When $m = 1$, the "rank condition" is simply $\nabla g_1(\mathbf{x}) \neq \mathbf{0}$; in general, it is the assertion that the gradient vectors $\nabla g_i(\mathbf{x})$ point in independent directions.

We now give the constrained version of the proposition above in the case of $m = 1$:

**Proposition.** *Let $Y \subset \mathbb{R}^n$ be a compact level set of the $C^1$ function $g$, and $f$ be a $C^1$ function that is defined in a neighborhood of $Y$. Then the maximum and minimum values of $f$ on $Y$ exist. They occur points where $\nabla f$ is a multiple of $\nabla g$, or where $\nabla g = \mathbf{0}$.*

*Proof.* Suppose that $\mathbf{a} \in Y$, and that $\nabla g \neq \mathbf{0}$ at (and hence near) $\mathbf{a}$. Using the Implicit Function Theorem, we can write one of the variables, say $x_n$, as $h(x_1, ..., x_{n-1})$ on $Y$ (near $\mathbf{a}$). Thus, we are seeking critical points for a function of $n-1$ variables, namely

$$F(x_1, ..., x_{n-1}) = f(x_1, ..., x_{n-1}, h(x_1, ..., x_{n-1})),$$

with $G(x_1, ..., x_{n-1}) = g(x_1, ..., x_{n-1}, h(x_1, ..., x_{n-1})) = 0$ determining the implicit differentiation of $h$. To be specific, writing $F_i$ for the partial derivative $F_{x_i}$, etc., we have for $i < n$ by the Chain Rule

$$(*) \quad F_i(x_1, ..., x_{n-1}) = f_i(x_1, ..., x_{n-1}, h(x_1, ..., x_{n-1})) + f_n(x_1, ..., x_{n-1}, h(x_1, ..., x_{n-1}))h_i(x_1, ..., x_{n-1})$$

and

$$(**) \quad G_i(x_1, ..., x_{n-1}) = g_i(x_1, ..., x_{n-1}, h(x_1, ..., x_{n-1})) + g_n(x_1, ..., x_{n-1}, h(x_1, ..., x_{n-1}))h_i(x_1, ..., x_{n-1}) = 0$$

Where $\nabla F = \mathbf{0}$ (*) and (**) have the same format. We write in condensed format: $f_i + f_n h_i = 0$ and $g_i + g_n h_i = 0$, and $g_n \neq 0$. This gives $f_i/f_n = g_i/g_n$ (when defined). Take it from here.

In the example with $f(x, y) = y - x^2$ and $g(x, y) = x^2 + y^2 - 1$, we are led to solve $\nabla(f - \lambda g) = 0$. Here, that's

$$-2x - \lambda(2x) = 0 \quad \text{and} \quad 1 - \lambda(2y) = 0,$$

which yields $x = 0$ or $\lambda = -1$, and $y = 1/2\lambda$; and, of course, $x^2 + y^2 = 1$. The solutions are $(x, y) = (0, \pm 1)$ and $(x, y) = (\pm\sqrt{3}/2, -1/2)$. The values of $f(x, y)$ at these points are $\pm 1$, and $-5/4$. We can see the absolute maximum and minimum values of $f$ on the unit circle.