

Some tips on the chain rule and max/min

1. The Chain Rule says that for the values of a C^1 function f in space on a curve, given parametrically by $\mathbf{x} = \mathbf{x}(t)$, i.e., $x = x(t)$, $y = y(t)$, $z = z(t)$,

$$\begin{aligned} (*) \quad \frac{d}{dt} f(x(t), y(t), z(t)) &= (\nabla f)(\mathbf{x}(t)) \bullet \mathbf{v}(t) \\ &= f_x(\mathbf{x}(t))x'(t) + f_y(\mathbf{x}(t))y'(t) + f_z(\mathbf{x}(t))z'(t), \end{aligned}$$

where $\mathbf{v}(t)$ is the velocity vector of the curve. In the above situation where there are instead *two* parameters (u, v) , (*) gives the partial derivative of f with respect to u (say) by simply changing $x'(t)$ to $\frac{\partial x}{\partial u}$, etc.

Could that be essentially all there is to the Chain Rule? Not quite. The Chain Rule is about differentiability, a notion that goes beyond partial derivatives. If the functions of (u, v) , x , y and z , are C^1 , then so is the composite with f . Then all are differentiable.

Also, remember that if f vector-valued, i.e., has more than one component function, the components may be treated separately (it's having several independent variables that causes "trouble" for the notion of differentiability).

2. The theory of local extrema of functions of two or more variables goes somewhat like the one variable case. One looks for the critical points (where $\nabla f = \mathbf{0}$), and tests it by the second derivative test:

In one variable, if $f'(t_0) = 0$, then $f''(t_0) < 0$ implies that f has a local maximum at t_0 ; and $f''(t_0) > 0$ implies that f has a local minimum. The remaining case, $f''(t_0) = 0$, is indeterminate, too delicate to call.

In two variables, if $\nabla f(\mathbf{x}_0) = \mathbf{0}$, then the two statements $\det Hf(\mathbf{x}_0) > 0$ and $f_{xx}(\mathbf{x}_0) < 0$ (from which it follows that $f_{yy}(\mathbf{x}_0) < 0$) imply that f has a local maximum at \mathbf{x}_0 ; $\det Hf(\mathbf{x}_0) > 0$ and $f_{xx}(\mathbf{x}_0) > 0$ (from which it follows that $f_{yy}(\mathbf{x}_0) > 0$) imply that f has a local minimum at \mathbf{x}_0 ; $\det Hf(\mathbf{x}_0) < 0$ implies that \mathbf{x}_0 is a saddle point for f , hence f has neither a local maximum nor a local minimum at \mathbf{x}_0 . The remaining case, where $\det Hf(\mathbf{x}_0) = 0$, is indeterminate, too delicate to call.