Outline of material on vectors

What follows is not complete, and it is admittedly very spotty. It is made solid by reading Chapters 1.1–1.5 of Colley. I am trying to touch the main conceptual points. Thus what I will write is *not everything*. You should ideally aspire to grasp it all just the same. I'll say something about this stuff in class tomorrow (2/7). But overall, this is the easiest material in the course, so I will not dwell on it. If you have questions, bring them up to me; there is no penalty for raising questions, as the course will be graded as described in the syllabus.

One last remark: concerning the function f in #2 from HW1, I was emphatic that to show that $f'(0)$ exists (c) you *must* use the definition of derivative. If you are still a little puzzled by that, just consider

$$g(x) = \begin{cases} x \text{ if } x \neq 0, \\ 0 \text{ if } x = 0, \end{cases} \quad \text{and} \quad h(x) = \begin{cases} x+1 \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

Moreover, you cannot decide continuity (a) without invoking the definition of continuity, so how can one expect *more*—diffentiability is stronger than continuity—by doing less?

Also, the point about making a substitution is more obvious in (a).

Vectors in $\mathbb{R}^n$, with $n = 2, 3$, or bigger (why not? You don't have to see things in order to deal with them!).

0. Read the syllabus if you haven't done so already! There's interesting material under *Academic Orientation* too.

1. Addition of two vectors via components (Def. (1.3)), or whole vectors (Fig. 1.6).

2. Multiplication by scalars (real numbers). This usually changes length and may reverse direction, but that's all.

3. Parametric equations in one independent variable. They make it visible that we are dealing with a *curve*. Allow here the stupid constant curves (all coordinates constant), whose image is just a point. Parametric equations are *never* unique. In particular, the choice of parameter is *never* unique. Don't feel that you are locked in by any particular parametrization!

4. Parametric equations for a line. These say "I go in [this] direction (specify a vector) and pass through [this] point." Note that there are many choices for the vector and the point. So you can never act as though there is only one.

5. Eliminating the parameter is like eliminating anything else. One gets two equations in three variables for a line in $\mathbb{R}^3$, with each equation giving a plane (see #10). This says what we might know or suspect: every line in space is the intersection of two planes. But the two planes are not uniquely determined ....

6. Two lines in the plane ($\mathbb{R}^2$) either coincide, are parallel, or intersect. There is a fourth possibility in $\mathbb{R}^3$: the lines can be *skew*, or equivalently, none of the above. A simple example of a pair of skew lines:
ℓ_1: $x = t$, $y = 0$, $z = 0$; ℓ_2: $x = 0$, $y = t$, $z = 1$.

7. Given two parametric curves in general, suppose they are written with the same symbol for the parameter. If you want to determine the intersection of the curves, you *must* acknowledge the non-uniqueness in #3, and rewrite them with *different* parameters. A stark example: the equations

$$(*) \qquad x = t, y = 0, z = 0 \quad \text{and} \quad x = t + 1, y = 0, z = 0$$

both parametrize the x-axis, yet x cannot be both t and $t + 1$ for the same value of t. If you imagine that t is time (why not), and (x, y, z) represents the position of a moving object, then equating the expressions in t for (x, y, z) is talking about *collision* between the objects, i.e., about passing through points *simultaneously.* Think of the two parametrizations in (*) as describing one person chasing another along the same line. Change the parameter to, say, u, in one of the parametrizations and then solve the equations for t <u>and</u> u, and it all works out.

8. Dot (or *inner*) product. In $\mathbb{R}^n$, this "product" takes two vectors and comes up with a <u>number</u> (also called a *scalar*). It determines length and detects perpendicularity. There is a formula with coordinates: $\mathbf{a} \bullet \mathbf{b} = \sum a_j b_j$, and one without $\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. It follows that you can determine angles between vectors from dot products.

9. In $\mathbb{R}^n$ suppose $\mathbf{a} \neq \mathbf{0}$, There is exactly one unit vector (i.e., of length 1) pointing in the direction of $\mathbf{a}$, and that is $\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|$. Given vectors $\mathbf{a}$ and $\mathbf{b}$, one can project $\mathbf{b}$ onto the direction of $\mathbf{a}$. How? Since the action takes place in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, we can draw this on a sheet of paper, making it simpler to visualize. Replacing $\mathbf{a}$ by $\mathbf{u}$ is a good first step (why?).

10. In the above there's a lot of talk about *planes*, so talking about them is overdue. In space a plane is determined by three points that don't all lie on one line (otherwise, they'd determine the line ... or even less).

To get an equation for a plane, the best thing to have is one point on the plane and a vector perpendicular to it (normal vector). As usual, neither is uniquely determined by the plane. The normal vector determines a family of parallel planes, and the one point will tell you which one of them it is. If $\mathbf{n}$ is the normal vector, and we notate its components as (a, b, c), the equation is $ax + by + cz = d$, where d is some constant. The given point can be written as (x_0, y_0, z_0). A point (x, y, z) is on the plane if and only if the displacement vector $\delta = (x - x_0, y - y_0, z - z_0)$, which is parallel to the plane, is, of course, perpendicular to the normal $\mathbf{n}$. This is detected by dot product (the magic number d is seen to be $ax_0 + by_0 + cz_0$). Conversely, a vector normal to the plane with equation $x - 2y + 3z = 4$ is readable from the equation: $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

11. Now, we can ask for parametric equations of a plane, or of the line perpendicular to the plane at a point on the plane, ...

12. And what's that thing called cross-product?