I challenge you to the dual

What's the big deal about dual spaces?

1. Let V be a vector space over the field $\mathbb{F}$. By definition, the *dual space* of V is

$$V^* = \mathcal{L}(V, \mathbb{F}),$$

the linear transformations from V to $\mathbb{F}$. We use as generic notation: for $\alpha \in V^*$, $\alpha : V \to \mathbb{F}$. We have seen that V^* is a vector space over $\mathbb{F}$.

2. When $V = \mathbb{F}^n$, when we talk about a linear mapping $S : \mathbb{F}^n \to \mathbb{F}^m$, it is given as L_A for some element A of $M_{m\times n}(\mathbb{F})$, the matrix representation with respect to the standard bases, if you like. Then, $(\mathbb{F}^n)^* \simeq M_{1\times n}(\mathbb{F})$.

3. When V is only isomorphic to $\mathbb{F}^n$ (that is, $\dim V = n$) we choose any basis β of V, and then we use the coordinate mapping $[\ ]_\beta : V \to \mathbb{F}^n$. Then one sees that $V^* \simeq (\mathbb{F}^n)^*$ (the isomorphism depends on β; see #4 below). We may thereby act as though V *is* $\mathbb{F}^n$

4. Let $T : V \to W$ be a linear transformation. There is an *induced* linear transformation $T^* : W^* \to V^*$ given by the following little diagram:

$$V \xrightarrow{T} W \xrightarrow{\alpha} \mathbb{F},$$

so $T^*(\alpha) = \alpha \circ T$.[1] When $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$, $(L_A)^* = L_{A^t}$.

5. There is a natural one-to-one linear mapping from V to $(V^*)^*$. It is defined without effort: each vector $\vec{v} \in V$ is mapped to $e_{\vec{v}}$ (evaluation at $\vec{v}$), i.e., $\alpha \mapsto \alpha(\vec{v})$. Check that this mapping is linear. It is an isomorphism if and only if V is finite-dimensional.

[1] For #3, one applies this for $T = [\ ]_\beta$ and $T = [\ ]_\beta^{-1}$