ETHICS PLEDGE: I agree to complete this exam without unauthorized assistance from any person or person’s work, materials or device.

Your name (print): ________________________________

Signature: ________________________________ Date: ________

INSTRUCTIONS: No books, no notes, no calculators! Write legibly, and show all relevant work—or risk losing credit. Your solutions will be judged as answers to questions asked.

If you need additional space to write your solutions to some problem, please use the back of the preceding page when possible. (It makes matters easier for everybody.)
1. Let $P_3(\mathbb{R})$ be, as usual, the linear space of polynomials of degree $\leq 3$ with real coefficients. Let $T : P_3(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation $T(f) = (f'(1), f''(0))$.

   a) Determine the rank of $T$.

   b) Determine the matrix representation of $T$, with respect to the usual bases of $P_3(\mathbb{R})$ and $\mathbb{R}^2$.

   c) For any subspace $W$ of $P_3(\mathbb{R})$, let $T_W$ denote the restriction of $T$ to $W$. Determine a 3-dimensional $W$ for which rank $T_W = \text{rank } T$, and one for which rank $T_W < \text{rank } T$. 
2. a) Let \( v_1, v_2, \ldots, v_
 \) be elements of a linear space \( V \). What is meant by a linear combination of \( v_1, v_2, \ldots, v_\)?

b) In terms of the linear combinations in part a), what is meant when one says that \( v_1, v_2, \ldots, v_\) are linearly independent?

c) Assume that \( V \) is finite-dimensional. Starting from the definition of \( [\ ]_\beta \), determine whether, for every isomorphism \( T : V \rightarrow V \) and every basis \( \beta \) of \( V \), \( [T^{-1}]_\beta = ([T]_\beta)^{-1} \).

d) Take \( V = \mathbb{R}^3 \), with standard basis \( \{e_1, e_2, e_3\} \). Let \( \beta = \{e_1 - e_3, e_1 + e_2 + e_3, 2e_1 + e_2 - e_3\} \). First, show that \( \beta \) is a basis of \( \mathbb{R}^3 \). Then determine the standard coordinates of the vector \( x \in \mathbb{R}^3 \) that satisfies \( [x]_\beta = e_1 + 2e_2 + 3e_3 \).
3. Let $W$ be the plane in $\mathbb{R}^4$ with equations $x_1 + 3x_2 - x_3 + x_4 = 0$, $x_1 + 2x_2 + x_3 - 2x_4 = 0$.
   a) Determine a basis of $W$.

b) Let $P : \mathbb{R}^4 \to \mathbb{R}^4$ be orthogonal projection onto $W$. Determine an orthonormal basis $\beta$ of $\mathbb{R}^4$ with the property that $[P]_\beta$ is diagonal, and give that diagonal matrix.

c) What is meant when one says that a subspace $W'$ of $\mathbb{R}^4$ is complementary to $W$? Also, show that $W' = \text{Span}\{e_1, e_2\}$ is complementary to $W$.

d) For $W'$ complementary to $W$, what is meant by the projection onto $W$ along $W'$ (denoted here by $P^W_W$)? For $W' = \text{Span}\{e_1, e_2\}$, determine the characteristic polynomial of $P^W_W$. 
4. a) What is meant when one says that two square matrices are \textit{similar}?

b) Show that similarity is an equivalence relation.

c) Take the six $3 \times 3$ matrices given below, and determine the similarity equivalence classes:

\[
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

Have a nice day!
[15] 5. Show that there is exactly one linear transformation $T : P_1(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ that satisfies

$$T(2 + 3t) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad T(3 + 4t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

and give an explicit formula for $T(p)$ ($p \in P_1$).

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[5,10,10,5,10,15,5] 6. Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional vector space over some field $\mathbb{F}$.

a) Let $W$ be a subspace of $V$. What is meant when one says that $W$ is $T$-invariant?

b) True or false: Every $T$-invariant subspace $W$ has a $T$-invariant complementary subspace. Explain.

c) What is meant by a generalized eigenspace of $T$. (Start with an eigenvalue of $T$.) Show that generalized eigenspaces are $T$-invariant.
6. (cont’d) Assume from now on that the characteristic polynomial \( f_T(t) \) of \( T \) splits.

   d) Specify a \( T \)-invariant complement for a generalized eigenspace of \( T \).

   e) On the road to Jordan, we were down to the following situation. Let \( W \) be a vector space on which a linear operator \( U : W \to W \) has splitting characteristic polynomial, with zero as the only eigenvalue. Show that \( U \) is nilpotent (i.e., \( U^m = 0 \) for some \( m \)). Indeed, show that \( U^d = 0 \), where \( d = \dim W \).

f) Show that \( W \) is cyclic if and only if the following equivalent statements hold:
   - i) \( U^{d-1} \neq 0 \);
   - ii) \( \dim N(U) = 1 \). (\( N(U) \) denotes the null space of \( U \)).

g) Give your sense of what the Jordan canonical form and its proof are saying, in 100 words or fewer. (What’s this, an essay question?)
[10] 6g' (Alternative to 6g). Let \( \dim V = n \), so \( V \) is isomorphic to \( \mathbb{F}^n \). Given the set of distinct eigenvalues of a matrix, specify the least amount of numerical data one needs in order to determine the Jordan canonical form of \( T \)? [The wording of this question may be a bit vague, but try to answer it anyway.]