Fulfilling Jordan’s legacy

A. Suppose an \( n \times n \) matrix \( A \) has an eigenvalue \( \mu \) with multiplicity, say, 3. It doesn’t matter how large \( n \) is (but \( n \geq 3 \)). One thing that the theory tells us: The generalized eigenspace \( K_\mu(A) \) has dimension 3, and \( K_\mu(A) = N((A - \mu I)^3) \). You can determine this subspace explicitly by just doing “what it says”! Solve the system of linear equations \((A - \mu I)^3 x = 0\) by old faithful (row reduction). This will pick out a basis for \( K_\mu(A) \).

Here’s a good way (maybe the best way) to proceed.

1. We know that there are basically three possibilities for the contribution of \( \mu \) to the canonical form (3 blocks of size 1; a block of size 1 and a block of size 2; one block of size 3).

2. Determine the number of blocks by computing the dimension of the actual eigenspace, \( E_\mu(A) = N((A - \mu I)) \). Since the first basis vector giving a block for the “\( \mu \)-part” of the canonical form satisfies the equation \((A - \mu I)x = 0\), the dimension \( d \) of \( E_\mu(A) \) gives the number of blocks. Thus, in the case under discussion it determines which case in \#1 is occurring.

Take each case for \( d \) separately.

a) The case \( d = 3 \) is easy to do completely (why?).

b) Actually, the case \( d = 1 \) is easy to do also: determine a basis vector \( x \) for the 1-dimensional \( E_\mu(A) \) and solve \((A - \mu I)^2 x = y \). They the \((A - \mu I)\)-cyclic subspace generated by \( y \), has basis vectors \( y, (A - \mu I)y, (A - \mu I)^2 y = x \). This is a basis giving the Jordan \( 3 \times 3 \) block.

c) So we have one more case, namely \( d = 2 \). We must find vectors \( x_1 \) and \( x_2 \) that give a basis for the 2-dimensional space \( E_\mu(A) \), such that \((A - \mu I)x = x_1 \) has no solution, while and \((A - \mu I)x = x_2 \) has a solution. Put another way, \( x_1 \) is chosen to span the 1-dimensional (why?) space \( R(A - \mu I) \cap E_\mu(A) \), and \( x_2 \) can be taken to be any vector in \( E_\mu(A) \) linearly independent of \( x_1 \).

d) I hope it’s clear that we have basically accomplished Mission Possible.

B. Let’s up the ante in A by supposing that the multiplicity of \( \mu \) is 4. The steps are analogous, the there are more of them.

1. We know that there are basically four possibilities for the contribution of \( \mu \) to the canonical form (4 blocks of size 1; two blocks of size 1 and a block of size 2; one block of size 1 and a block of size 3; two blocks of size 2).

2. Determine the number of blocks by computing the dimension of the actual eigenspace, \( E_\mu(A) = N((A - \mu I)) \). Since the first basis vector giving a block for the “\( \mu \)-part” of the canonical form satisfies the equation \((A - \mu I)x = 0\), the dimension
$d$ of $E_\mu(A)$ gives the number of blocks. Thus, in the case under discussion $d$ almost determines which case in $\#1$ is occurring.

Take each case for $d$ separately.

a) The case $d = 4$ is easy to do completely.

b) Actually, the case $d = 1$ is easy to do also: determine a basis vector $x$ for the $1$-dimensional $E_\mu(A)$, Solve $(A - \mu I)^3 x = y$, and let $y$ generate a cyclic subspace for $(A - \mu I)$. Continuing as in A(2(b)), we get a basis giving the Jordan $4 \times 4$ block.

c) The case $d = 3$ is now straightforward, for it goes as in A(2(c)) above: determine $R(A - \mu I) \cap E_\mu(A)$. Thus, ....

d) So we have one more case, namely $d = 2$. The novelty item this time is that there are two possibilities for the contribution of $\mu$ to the canonical form (one block of size 1 and a block of size 3; two blocks of size 2). How do we distinguish them? Reason it out. In the first case, we would have a non-zero vector $v \in R((A - \mu I)^2) \cap E_\mu(A)$, so adapt A(2(c)). In the second case, we wouldn’t(!), so pick any basis $\{x_1, x_2\}$ for $E_\mu(A)$ and solve $(A - \mu I)y_i = x_i$ for each $i = 1, 2$.

e) I hope it’s clear that we have basically accomplished Mission More-Complicated-but-Possible.

C. Carry out the appropriate procedure for each eigenvalue, and Viola! Of course, the mechanics of the process get more intricate as $d$ increases.

On the Final, you will be asked to carry out something of the above sort. The calculations described above, e.g., solve ..., must be done explicitly.