Echelon

We were at the following stage:

For any $m \times n$ matrix $A$ (with entries in the field $\mathbb{F}$), we can write as $A = PR$, where $P$ is an invertible $m \times m$ matrix and $R$ is an $m \times n$ reduced row-echelon matrix. We saw that that the rank of $A$ and the rank of $A^t$ are equal, deducing this from the same for $R$, where it is “obvious”.

More is true. Let $j(i)$ be the (increasing) function that gives the column of $R$ that has the $i$-th pivotal 1, for all $i$ satisfying $1 \leq i \leq r$, with $r$ denoting the rank of $R$ (equaling the rank of $A$, by the above). Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{F}^n$, and $\{f_1, ..., f_m\}$ the standard basis of $\mathbb{F}^m$. The correct statement about $R$ is that

\[(*)\quad R(e_{j(i)}) = f_i\]

for $1 \leq i \leq r$, i.e., the $j(i)$ column vector of $R$ is $f_i$. Applying $P$ to $(*)$, we get

\[(**)\quad A(e_{j(i)}) = P(f_i)\]

As the $f_i$’s are linearly independent, the $P(f_i)$’s are also linearly independent (why?). Thus, the $A(e_{j(i)})$’s are linearly independent, so these are $r$ vectors that form a basis of $\mathcal{R}(A)$. We should know by now that these vectors are just the $j(i)$-th column vectors of $A$, the same columns that have the $f_i$’s in $R$. 