ETHICS PLEDGE: I agree to complete this exam without unauthorized assistance from any person or person’s work, materials or device.

Your name (print): ________________________  Section: ________

Signature: ________________________________  Date: ________
No books, no notes, no calculators or other electronic devices. Write legibly, and show all relevant work—or risk losing credit. Answer what is asked, and only what is asked.

[5, 10] 1. This problem is about the linear system:

\[ x_1 + x_2 = 1, \quad x_1 - x_2 = 0, \quad 2x_1 + x_2 = 0. \]

a) Determine that there are no solutions to this system.

b) The least-squares (approximate) solution of the system is the exact solution for a system with the same left-hand sides, but different constants on the right-hand sides. Determine both those constants and the least-squares solution.

[10] 2. Are the linear spaces \( P_3 \) and \( \mathbb{R}^{2 \times 2} \) isomorphic? \textbf{Explain.}
3. a) Determine all unit vectors that are perpendicular to the plane in $\mathbb{R}^3$ (whose equation is) $x_1 + 2x_2 - 3x_3 = 0$.

b) Determine an orthonormal basis \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} of $\mathbb{R}^3$ for which \{\vec{u}_1, \vec{u}_2\} is a basis for the plane $x_1 + 2x_2 - 3x_3 = 0$.

4. This problem is about functions $T: V \rightarrow W$, when $V$ and $W$ are linear spaces.

a) What is meant when one says “$T$ is linear”? That is, give the definition of linear.

b) Let $V = \mathbb{R}^{2\times 2}$, $W = \mathbb{R}$. Explain in terms of a) why the function $T(A) = \det(A)$ is not linear.
[15] 5. Let $A$ be the $4 \times 4$ matrix
\[
\begin{bmatrix}
0 & 3 & 6 & 4 \\
0 & 4 & 6 & 3 \\
0 & 0 & 0 & 2 \\
2 & 5 & 5 & 1
\end{bmatrix}.
\]

a) Calculate the determinant of $A$.

b) Determine whether $A^5$ is invertible.

[20] 6. We define a linear transformation $\Phi : P_3 \to \mathbb{R}^2$ by $\Phi(f) = f(0)\mathbf{e}_1 + f(1)\mathbf{e}_2$.

a) Show that $\Phi$ is, in fact, linear.

b) Define the rank of a linear transformation $T$. Determine directly that the rank of $\Phi$ is 2.

c) Define the nullity of a linear transformation $T$.

d) Complete this statement of the rank-nullity formula, and use it to determine the nullity of $\Phi$. Let $T : V \to W$ be a linear transformation of finite-dimensional linear spaces. Then ...
[15] 7. In $\mathbb{R}^4$, let $\vec{v}_1 = \vec{e}_1$, $\vec{v}_2 = 9\vec{e}_1 + 10\vec{e}_2$, $\vec{v}_3 = 13\vec{e}_1 - 12\vec{e}_2 + \vec{e}_3$, $\vec{v}_4 = 11\vec{e}_1 + 8\vec{e}_2 - 7\vec{e}_3 + 14\vec{e}_4$. Determine whether $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis of $\mathbb{R}^4$.

[10] 8. An inner product on continuous functions on $[0, 2\pi]$ is defined by

$$<f, g> = \int_0^{2\pi} f(t) g(t) \, dt.$$  

a) Show that the functions $\sin t$ and $\cos t$ on $[0, 2\pi]$ are orthogonal.

b) Show that with respect to the inner product on continuous functions on $[0, 1]$, namely

$$<f, g> = \int_0^1 f(t) g(t) \, dt,$$

$\sin t$ and $\cos t$ on $[0, 1]$ are not orthogonal.