A tale of two bases

Let's start with a tale of one basis of \( \mathbb{R}^n \). Get down to the bottom of things: let \( n = 2 \). I hope you see that \( n = 1 \) is too easy for most things, but not here! So let \( n = 1 \).

If I told you that we had a Calc 0 function \( f : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \), and that \( f(1) = 2 \); determine \( f \). You'd think I was toying with you. You'd say, there are infinitely many such functions, e.g., \( f(x) = 2x^k \) for every \( k > 0 \). And that's absolutely correct. But if I added that \( f \) is linear, the only answer would be \( f(x) = 2x \).

The message you should have picked up by now is the notion of linearity, as I put it. That's Fact 1.3.9 from our book. It follows that a linear mapping is completely determined by its value on any one basis. How? Let \( B = \{ \vec{v}_1, \ldots, \vec{v}_n \} \) be a basis of \( \mathbb{R}^n \). Write \( \vec{v} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n \). The \( c_i \)'s are determined uniquely by \( \vec{v} \) (that's what a basis is all about), then

\[
(*) \quad T(\vec{v}) = c_1 T(\vec{v}_1) + \ldots + c_n T(\vec{v}_n).
\]

There are simple rules for working with linear transformations. Never forget that! Admittedly, the task of finding the actual \( c_i \)'s for a given vector \( \vec{v} \) (and given \( B \), of course) involves solving a certain system of linear equations, but we should know how to do that by now.

Since \( \{1\} \) is a basis of \( \mathbb{R}^1 \), meaning that every number \( x \) is a multiple of 1, linearity gives \( f(x) = f(x \cdot 1) = xf(1) \); \( f(1) = 2 \) thereby determines \( f(x) = 2x \).

Let's move onward to \( \mathbb{R}^2 \) and functions of two variables. Suppose I have a secret function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and I told you (say) that \( f(1, 1) = 2 \) and \( f(1, -2) = 3 \). Tell me my function or you die! That would reduce attendance in lecture drastically. But if I told you the wonderful feature that \( f \) is linear, there is sufficient information; there is exactly one such linear \( f \). Why? Because \( B = \{ (\vec{e}_1 + \vec{e}_2), (\vec{e}_1 - 2\vec{e}_2) \} \) is a basis for \( \mathbb{R}^2 \) (explain). Linearity, in its version (*) , tells you how to deduce the value of \( f \) at every vector of \( \mathbb{R}^2 \). [Do it.]

The above contains the essential issues of Linear Algebra, and we have not gotten beyond \( \mathbb{R}^2 \) as domain and \( \mathbb{R}^3 \) as codomain!

Here comes the tale of two bases. From A to Z: A is a student, Z is a mathematics instructor. [The comments in brackets are not spoken, and they probably wouldn't occur at JHU.]
- Z: Which vector in $\mathbb{R}^2$ has coordinates 1,0?
- A: $[\vec{e}_1$ of course.]

Don’t you have to specify a basis?
- Z: Yes, of course. The basis I have in mind is $B = \{(\vec{e}_1 + \vec{e}_2), (\vec{e}_1 - 2\vec{e}_2)\}$.
- A: Why that one?
- Z: [Do you need a reason?]

It will become clear later on. I have something up my sleeve.
- A: OK. The answer is the first element of your [stupid] basis, namely $\vec{e}_1 + \vec{e}_2$.
- Z: Good, now you’re getting serious. Next, tell me the matrix $T$ of the coordinate mapping with respect to the basis $B$.
- A: OK. Isn’t that the one where you put the vectors of $B$ in order as the columns of $T$:

\[
(*) \quad T = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}
\]

[*#*!] Or did I give $T^{-1}$? ....
- Z: Stay calm. The matrix you gave takes $\vec{e}_1$ to $\vec{e}_1 + \vec{e}_2$, and $\vec{e}_2$ to $\vec{e}_1 - 2\vec{e}_2$. That’s backwards, right? The $B$-coordinate map takes any vector in $\mathbb{R}^2$ to the coefficients of the vectors in the ordered basis under consideration. You really want $\vec{e}_1 + \vec{e}_2$ to map to $\vec{e}_1$, etc. So you have written down the inverse matrix. Can you invert a $2 \times 2$ matrix?
- A: Do you take me for a dummy?
- Z: [No comment.] Of course not. Do you see the point of linearity? If you know you have a linear transformation of $\mathbb{R}^m$ (into $\mathbb{R}^n$), it is completely specified by its value on $m$ linearly independent vectors of $\mathbb{R}^m$. Just take linear combos and invoke linearity. (Conceptually, you can think that without carrying it out in numbers. Do you really want to carry that out in (say) $\mathbb{R}^{30}$?)
- A: I think I’m getting the point.
- Z: Great! Now, let’s determine the matrix of $T_A$, where $A$ is the $2 \times 2$ matrix:

\[
A = \frac{1}{3} \begin{bmatrix} 13 & 2 \\ 4 & 11 \end{bmatrix}
\]

- A: It’s $A$, of course.
- Z: That’s for the standard basis. I’ve been insisting on my basis $B$. You’ll see why.

We go straight to the ending. The matrix is

\[
B = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}
\]
As the theory produces unambiguously, $B = TAT^{-1} (S^{-1}AS$, with $S = T^{-1}$).

[Check it out.] The punch line: that mess of a matrix $A$ is describing a transformation that stretches by a factor of 5 in one direction (of $\vec{v}_1 = \vec{e}_1 + \vec{e}_2$), and by a factor of 3 in another ($\vec{v}_2 = \vec{e}_1 - 2\vec{e}_2$). So what if these directions aren’t perpendicular! Isn’t that a clearer picture of what $T_A$ does than you get by just looking at $A$?

There is a procedure for making transformations look simpler, and it comes later in the course.