Reflection on change of matrix

Let $A$ be an $n \times n$ matrix. The words *change of matrix* in the title could refer to what happens when you describe the linear transformation $T_A$ with respect to some basis $\mathcal{B}$ other than the standard basis $\mathcal{A}$. You should first appreciate the statement

$$[T_A]_{\mathcal{A}} = A,$$

and that it says how $T_A$ acts in terms of $\mathcal{A}$-coordinates.

Next, we understand that the $\mathcal{B}$-matrix $[T_A]_{\mathcal{B}}$ can be described in two ways:

a) The matrix $[T_A]_{\mathcal{B}}$ is the one and only matrix that satisfies:

$$[A\bar{v}]_{\mathcal{B}} = [T_A(\bar{v})]_{\mathcal{B}} = [T_A]_{\mathcal{B}}[\bar{v}]_{\mathcal{B}}.$$

Once we accept that all bases of $\mathbb{R}^n$ are equal in the eyes of the law, (2) has the same content as (1). It coincides with (1) when $\mathcal{B} = \mathcal{A}$, and it says how $T_A$ acts in terms of $\mathcal{B}$-coordinates.

b) Since we are presumed to be given the matrix $A$, there is the well-known formula that minimizes the brain pain:

$$[T_A]_{\mathcal{B}} = S^{-1}[T_A]_{\mathcal{A}} S = S^{-1}AS,$$

where $S$ is the matrix whose column vectors are the vectors in $\mathcal{B}$.

Note that (3) says, *once you know the vectors in $\mathcal{B}$, and how to multiply and invert $n \times n$ matrices, computing the $\mathcal{B}$-matrix of $T_A$ is completely mechanical.* A good linear algebra student will understand a) as well.

However, the above is not what I had in mind. I want to point out some very simple maneuvers concerning matrices and their eigenvalues.

i) $\lambda$ is an eigenvalue of $A$ if and only if $\lambda + c$ is an eigenvalue of $A + cI$. Indeed we have the following elementary calculation

$$A\bar{v} = \lambda \bar{v} \quad \text{if and only if} \quad (A + cI)\bar{v} = (\lambda + c)\bar{v},$$

from which it follows that there is even an *equality* of eigenspaces

$$\ker(A - \lambda I) = E_\lambda(A) = E_{\lambda+c}(A + cI).$$

ii) Similarly,

$$E_\lambda(A) = E_{k\lambda}(kA).$$