Lest we’ve forgotten

I said in lecture today that the solution set of $A\vec{x} = \vec{0}$ (a homogeneous system because of the $\vec{0}$ on the right-hand side) is given by our elimination process as the span of a specific set of vectors. Again, for given vectors $\vec{v}_1, \ldots, \vec{v}_t$, the span of these vectors is the set of all linear combinations of these vectors, namely all vectors that can be written in the form

$$t_1 \vec{v}_1 + \ldots + t_t \vec{v}_t,$$

where $t_1, \ldots, t_t$ are scalars (i.e., $\in \mathbb{R}$).

Let’s illustrate the above by taking a particular linear system. We may as well take the coefficient matrix to be a reduced row-echelon matrix. Why? Take as the matrix $A$

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

Then $A\vec{x} = \vec{0}$ has solution: $x_1 = -2t$, $x_2 = t_1$ (parameter for the free variable $x_2$), $x_3 = 0$, $x_4 = t_2$ (parameter for the free variable $x_4$). In our preferred vector form, that’s

$$\vec{x} = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where $t_1$ and $t_2$ are arbitrary real numbers. The above is the span of those two vectors, giving the solution set to $A\vec{x} = \vec{0}$, which we are calling ker $A$. Thinking of what the example represents, we can see that the kernel of any matrix is equal to the span of a finite number of vectors (possibly zero), namely the number of free variables.

As far as seeing that the image of a matrix, namely the span of its column vectors, is the solution set of some system of linear equations, we will address that later. Soon, but later!