Basis for a span

The issue I wish to address here is the question: Given a set of $\ell$ vectors in $\mathbb{R}^n$

$$S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_\ell\},$$

find a basis of $\text{Span}(S)$. One is asking, in effect, to discard unnecessary (i.e., redundant) vectors.

There are three subquestions, which will be answered in turn:

a) if you want to use only a subset of the given set of vectors, there's a rule that may seem like magic, or even nonsense:

1. Make an $n \times \ell$ matrix $A$ with $S$ as column vectors.
2. Reduce $A$ to the reduced row-echelon matrix $\text{rref}(A)$.
3. Take the columns of $A$ corresponding to those columns of $\text{rref}(A)$ where the leading 1's lie. These vectors give a basis of $\text{Span}(S)$.

The mystery in the above lies in the fact that row operations usually change the columns of a matrix. (We will proceed differently in (b) below, where we convert our column vectors to row vectors, and then convert back at the end.) A decent way to view this is that row reduction is accomplished by the left-multiplication $A \mapsto EA$, where $E$ is invertible. The reverse process goes likewise, namely by left-multiplication by $E^{-1}$. (We've touched upon this; see Ch. 2.4, #50–53.) Can you see the process and its inverse take linearly independent sets to linearly independent sets, spans to spans?\(^1\) We then note that $\text{im}(\text{rref}(A))$ has an obvious basis, namely the columns with the leading 1's. Applying $E^{-1}$ to them justifies step #3.

b) if you are willing to take any basis;

Lay the (column) vectors in $S$ as the rows of an $\ell \times n$ matrix; call it $B$. Do row operations on $B$ to get it into the reduced row-echelon matrix $\text{rref}(B)$. The span of the rows of $\text{rref}(B)$ and those of $B$ coincide. The non-zero rows, set as column vectors, form a basis for $\text{Span}(S)$.

c) if you want to get a basis for the image of an $n \times \ell$ matrix $A$.

This is easy now. Take $S$ to be the set of column vectors of $A$, and do your favorite of a) or b).

\(^1\)That was easy to say, but let's state it precisely. Let $B$ be an invertible $n \times n$ matrix. Then a set of vectors $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_\ell\}$ in $\mathbb{R}^n$ is linearly independent if and only if $\{B(\vec{v}_1), B(\vec{v}_2), ..., B(\vec{v}_\ell)\}$ is linearly independent. Likewise, a set of vectors $S$ spans a subspace $W \subseteq \mathbb{R}^n$ if and only if $\{B(\vec{v}_1), B(\vec{v}_2), ..., B(\vec{v}_\ell)\}$ spans $B(W)$ (the image of $W$ under $B$).