Metric Spaces Worksheet 3

Sequences II

We’re about to state an important fact about convergent sequences in metric spaces which justifies our use of the notation \( \lim a_n = a \) earlier, but before we do that we need a result about M2 – the separation axiom.

**Lemma 1** (only equal points are arbitrarily close). If \((X, d)\) is a metric space and the points \(x, y \in X\) satisfy \(\forall \varepsilon \in [0, \infty), [d(x, y) < \varepsilon]\), then \(x = y\).

**Complete the proof here**
**Theorem 2** (limits are unique). In a metric space \((X, d)\), if \((a_n)\) is a convergent sequence and 
\((a_n) \to a\) and \((a_n) \to a'\) then \(a = a'\).

**Hint 3.** In order to prove this you should:

1. use the triangle inequality (M4) to usefully write \(d(a, a') \leq \text{something},\)

2. apply lemma 1.

*Complete the proof here*
With our knowledge of convergence we can tie together our intuitions about eventually constant sequences and convergence.

**Lemma 4** (eventually constant is convergent). *If \((X,d)\) is a metric space and \((a_n)\) is an eventually constant sequence in \((X,d)\), then \((a_n)\) converges.*

*Complete the proof here*
The converse of this is not generally true.

**Question 5.** Can you find a convergent sequence which is not eventually constant?

*Complete the proof here*

There is, however, a case in which it is true.

**Lemma 6** (convergent is constant in discrete). *If* \((X,d)\) *is a discrete metric space then every convergent sequence is eventually constant.*

**Hint 7.** What happens when \(0 < \epsilon < 1\) in a discrete metric space?

*Complete the proof here*
Sometimes one comes across sequences which would converge if they could, but the point to which they would converge is not in the ambient metric space. Such a sequence does not satisfy the definition of convergence, but we do have a mathematical way of describing this situation.

**Definition 8** (Cauchy sequence). A sequence \((a_n)\) in a metric space \((X, d)\) is called a **Cauchy sequence** when it satisfies

\[
\forall \varepsilon \in [0, \infty), \exists N \in \mathbb{N}, \forall n, m \in \mathbb{N}, [(n \geq N) \land (m \geq N) \rightarrow d(a_n, a_m) < \varepsilon].
\]

We think of these as “almost convergent” sequences, and, as we would hope, “almost convergent” is implied by actual convergence.

**Lemma 9** (convergent is Cauchy). If \((a_n)\) is a convergent sequence in a metric space \((X, d)\) then \((a_n)\) is Cauchy.
The following lemma explains the intuition that Cauchy sequences “would converge if they could, but sometimes the point to which they would converge is not in the ambient metric space.”

**Lemma 10.** Let \((Y, d)\) be a sub-metric-space of a metric space \((X, d)\). Suppose \((a_n)\) is a sequence in \((Y, d)\) that converges in \((X, d)\) to a point \(a \in X \setminus Y\), that is, \(a\) lies in \(X\) but not in the subspace \(Y\). Then:

1. The sequence \((a_n)\) is a Cauchy sequence in \((Y, d)\).
2. The sequence \((a_n)\) does not converge \((Y, d)\).

**Hint 11.** When proving this result, consider the following:

1. Recall that we defined a sequence in a subset of a metric space to be a sequence in the metric space whose every term is in the subset. Thus in particular, any sequence in a sub-metric-space is also a sequence in the ambient metric space.
2. For part 2, if \((a_n)\) converged to a point \(b\) in the metric space \(Y\), what could we say about the relationship between the points \(a\) and \(b\)?

*Complete the proof here*
We have already suggested that not every Cauchy sequence is actually convergent. In a sense suggested by the examples below, Cauchy sequences which are not convergent detect ‘holes’ in our metric space.

**Example 12 (Cauchy is not necessarily convergent)**

For both of these examples, prove that the sequence is Cauchy but not convergent.

1. Let \((a_n)\) be the sequence defined by \(a_n \equiv n/n + 1\), in the set \((-\infty, 1) \cup (1, \infty) \subseteq \mathbb{R}\) with the Euclidean metric. The first few terms are 0, 1, 2, 3, 4, 5, ...

2. Let \((b_n)\) be the sequence defined by \(b_n \equiv \text{the } (n+1)-\text{digit decimal expansion of } \sqrt{2}\), in the set \(\mathbb{Q} \subseteq \mathbb{R}\) with the Euclidean metric. The first few terms are 1, 1.4, 1.41, 1.414, or 1, \(\frac{14}{100}, \frac{141}{1000}, \frac{1414}{10000}, \ldots\)

*Complete the proof here*
Finally, it will prove useful later to understand how Cauchy sequences are sensitive to the finiteness of the space.

**Lemma 13** (Cauchy sequence in finite space is eventually constant). Let \((F,d)\) be a metric space, with \(F\) a finite set, and let \((a_n)\) be a Cauchy sequence in \(F\). Then \((a_n)\) is eventually constant.

**Corollary 14** (Convergent sequence in finite space is eventually constant). Every convergent sequence in a finite metric space is eventually constant.