Towards proof relevant category theory, as modelled by globular T-categories

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(not so proudly)

JHU

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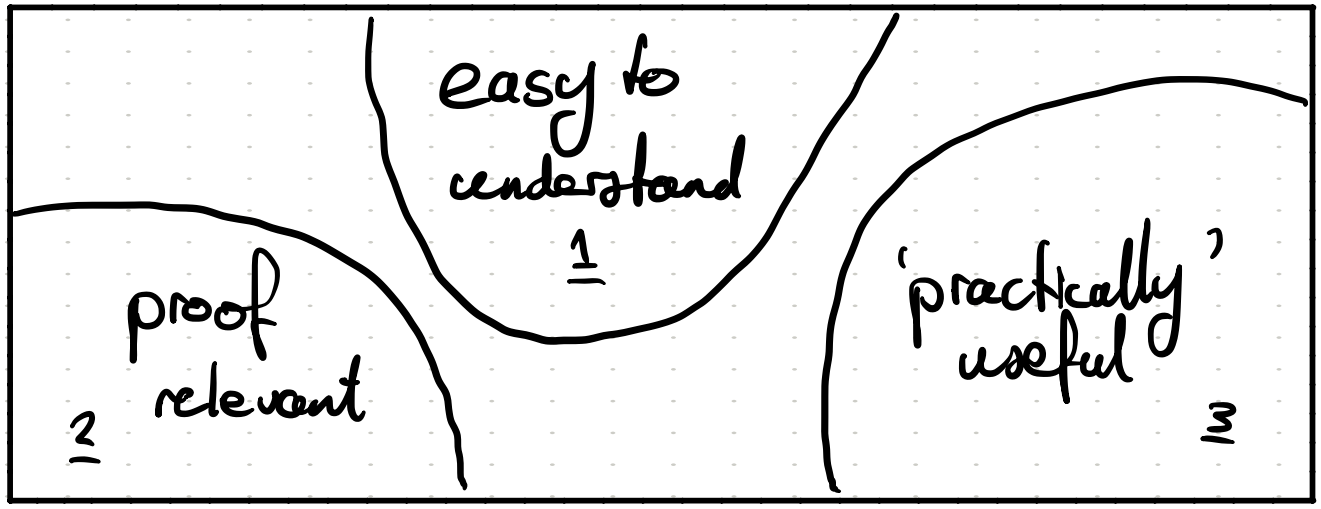
The state of higher category theory



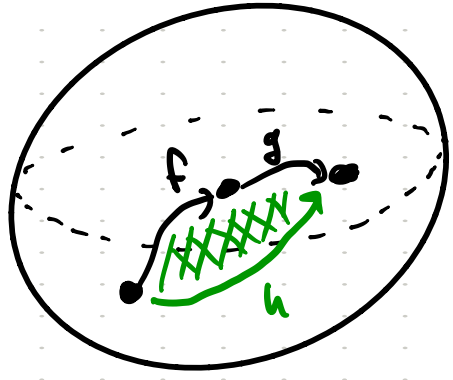
This talk: maybe it's a torus?

strict ω-categories $\in \underline{1}$ ⇢ (weaken?) ⇢ fundamental ?-groupoid $\in \underline{3}$ (types)

??? $\in \underline{2}$

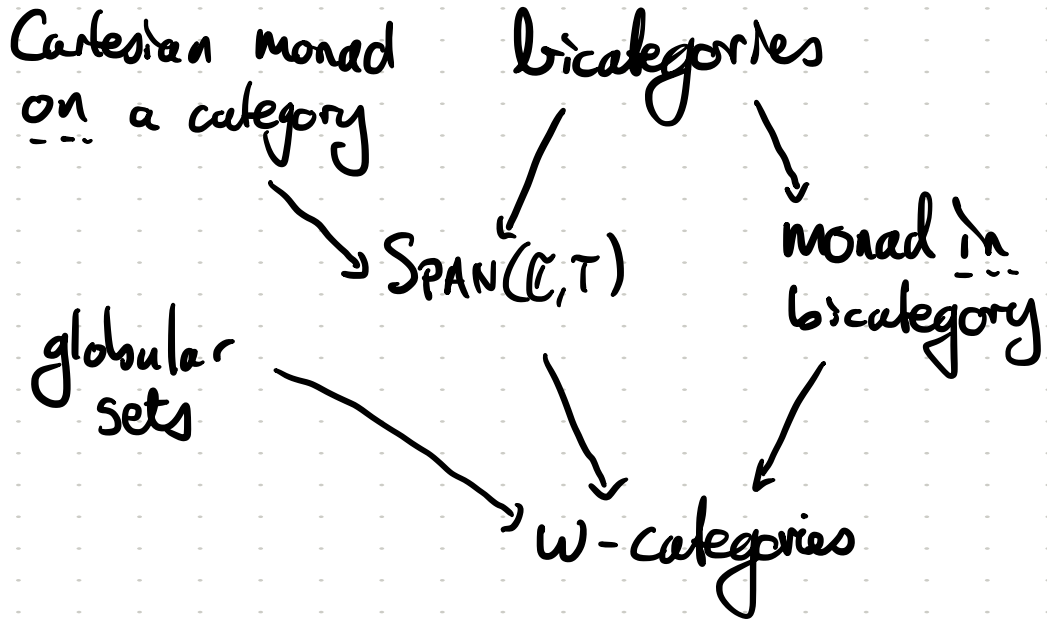
Proof relevance:

Consider fundamental 1-groupoid ...



the concatenation gf may be interpreted as h by ▨ but there is another proof ...

THE BIG PICTU

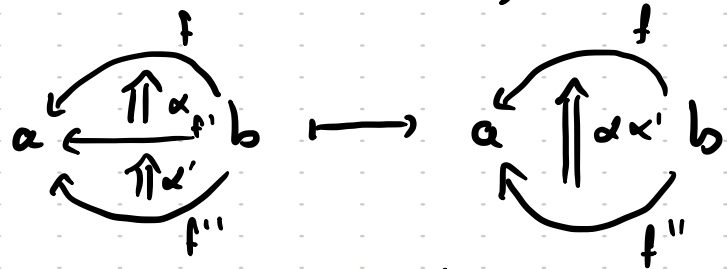


Background

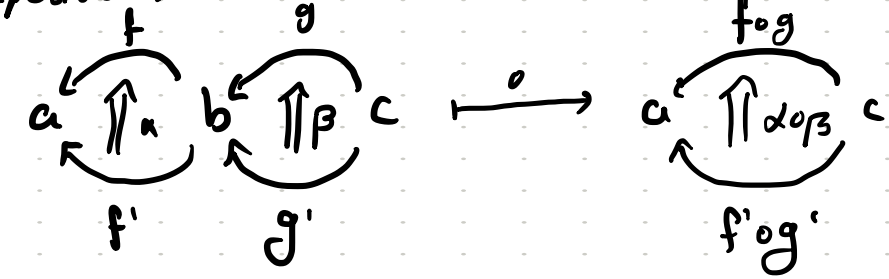
"Definition": A classical bicategory $\mathcal{K}$ consists:

- 0-cells
  "objects"
  $a, b, c, \dots$
- 1-cells
  "morphisms"
  $a \xleftarrow{f} b \in \mathcal{K}(b,a)$
- 2-cells
  " ? "
  $a \overset{f}{\underset{f'}{\Uparrow \alpha}} b \in \mathcal{K}(b,a)(f',f)$

- the structure of a category on each $\mathcal{K}(b,a)$
  "vertical" composition:

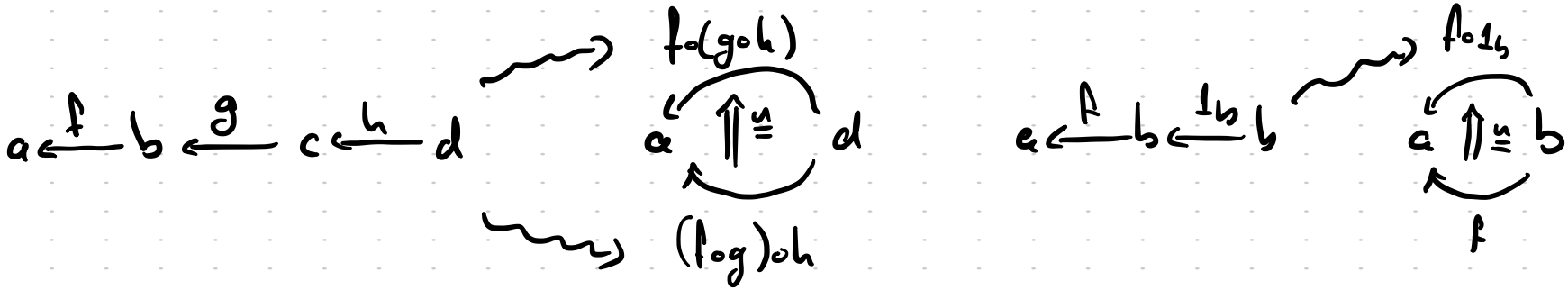


- composition functors $\mathcal{K}(b,a) \times \mathcal{K}(c,b) \xrightarrow{\circ} \mathcal{K}(b,a)$
  "horizontal" composition:



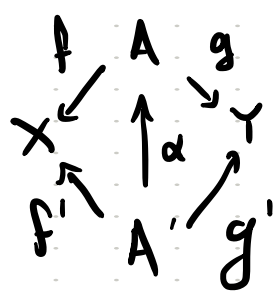
- specified "horizontal" units: $1_a \in \mathbb{K}(a,a)$

whose the composition is 'weakly' associative and unital.

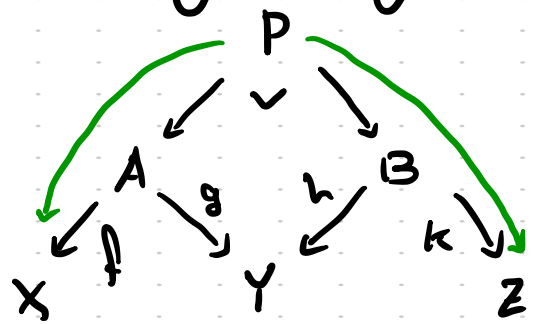


--- and these 2-cells themselves are subject to further coherences. (essentially the horizontal categorification of monoidal categories).

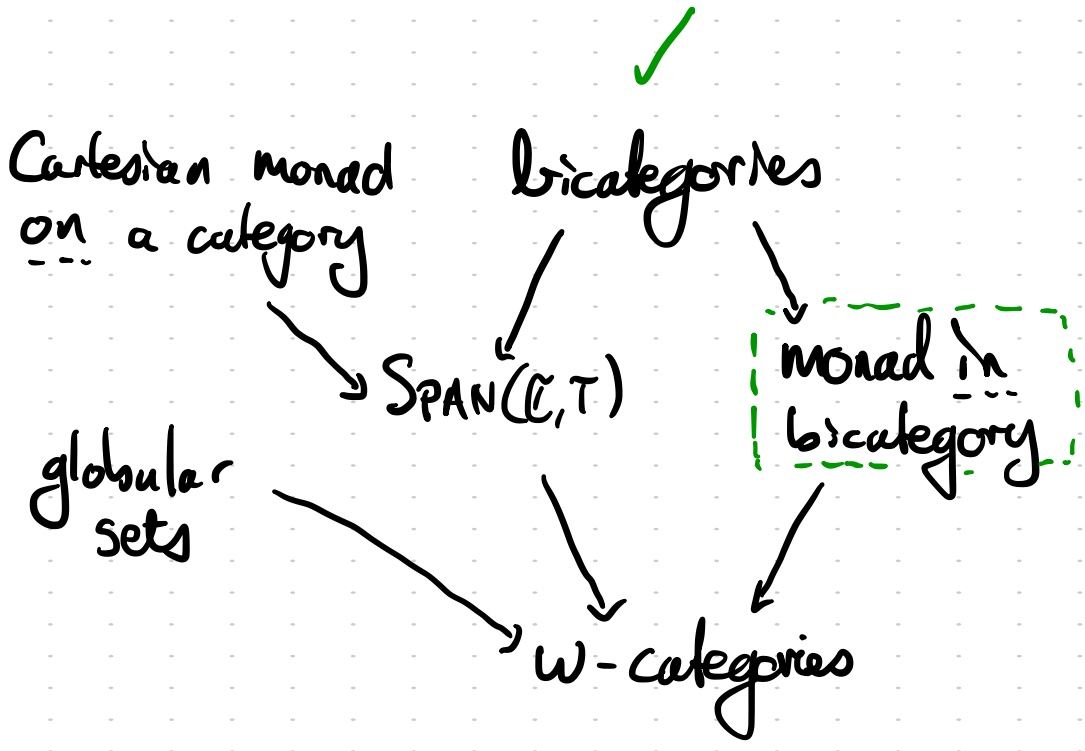
The only important example: $\mathrm{SPAN}(\mathrm{SET}, \mathrm{Id})$

- 0-cells are sets
- 1-cells are spans $X \xleftarrow{f} A \xrightarrow{g} Y$ of sets
  (f, g are set functions)
- 2-cells  are functions rendering commutative the diagram

  $\rightsquigarrow \mathrm{SPAN}(\mathrm{SET}, \mathrm{Id})(X, Y)$ is a category
- horizontal composition is given by pullback:

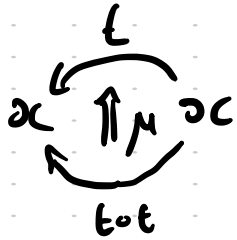
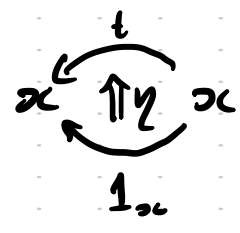


we will return to this in slightly more detail momentarily


✓
Cartesian monad on a category
bicategories
Span(ℂ,T)
monad in bicategory
globular sets
ω-categories

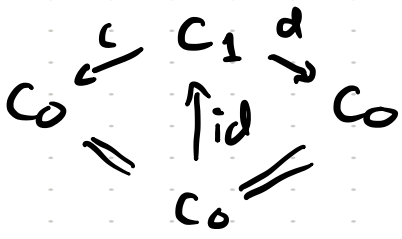
but now for something completely different:

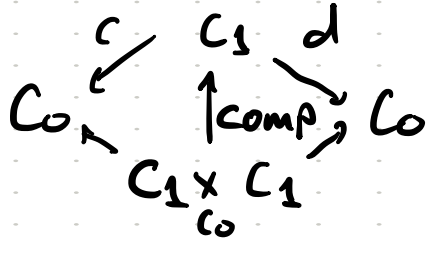
"Definition": A monad in a classical bicategory comprises

- a 0-cell x
- a 1-cell $x \xleftarrow{t} x$
- 2-cells  and 

... such that the 'monad laws' are satisfied.

Example: A monad in $\mathrm{SPAN}(\mathrm{SET}, \mathrm{Id})$ is ...

- a set C_0
- a function 

- a 1-cell $C_0 \xleftarrow{c} C_1 \xrightarrow{d} C_0$
- a function 

satisfying ...

seem familiar?

The insight of Burroni:

a category with pullbacks — a monad (of sorts) — parameters

a monad in $\mathrm{SPAN}(\mathrm{SET}, \mathrm{Id})$

is a category

Definition: A monad (T, μ, η) on a category $\mathbb{C}$ is cartesian just when T preserves pullbacks and the naturality squares of η and μ are pullbacks.

Examples:

1) $(\mathrm{Id}, \mathrm{id}, \mathrm{id})$, the identity monad

2) On SET,
   - $X \mapsto \sum(n : \mathbb{N}),\ X^n$ (free monoid)
   - $X \mapsto 1 + X$ (maybe monad)
   - $X \mapsto M \times X$ where M is a monoid

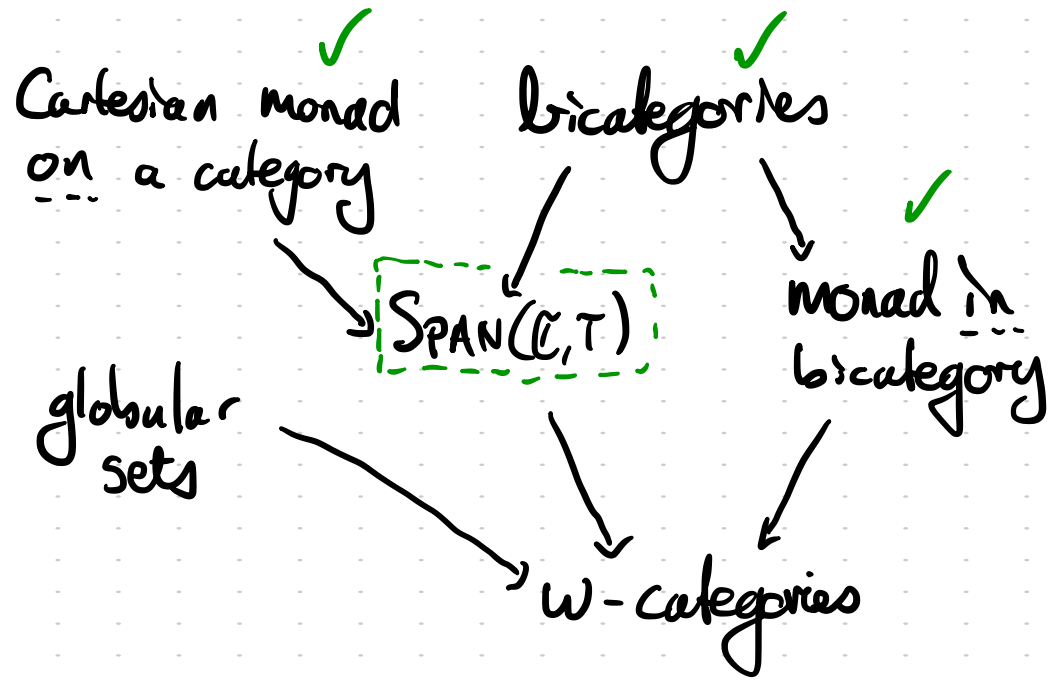
(Cartesian monad examples cont'd.)

3) On $\widehat{\mathbb{G}_2} := [\mathbb{G}_2^{op}, \mathrm{SET}]$, the free category monad

where $\mathbb{G}_2 := \{0 \rightrightarrows 1\}$

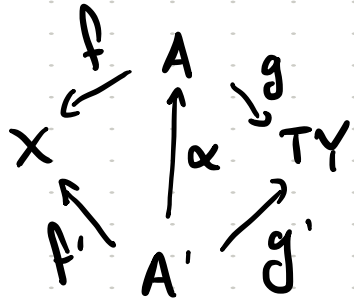
4) Any strongly regular theory

... abundant!



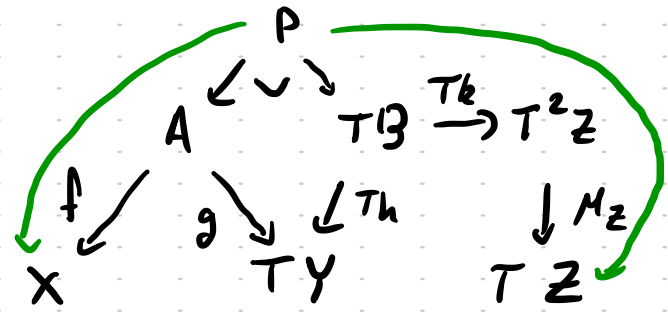
"**Definition**": For T a cartesian monad on $\mathbb{C}$ a category with chosen binary pullbacks the classical bicategory $\underline{\text{SPAN}(\mathbb{C}, T)}$ is given by

- 0-cells objects of $\mathbb{C}$
- 1-cells $X \leftarrow Y$ spans in $\mathbb{C}$ of the form $X \xleftarrow{f} A \xrightarrow{g} TY$
- 2-cells are morphisms



- composition is

$$X \xleftarrow{f} A \xrightarrow{g} TY \circ Y \xleftarrow{h} B \xrightarrow{k} TZ$$



- horizontal identities are

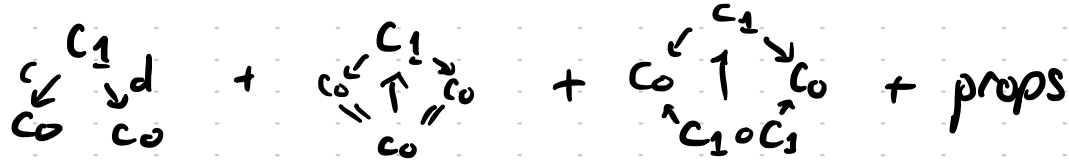
$$A = A \xrightarrow{\eta_A} TA$$

NB: would be Kleisli composition for

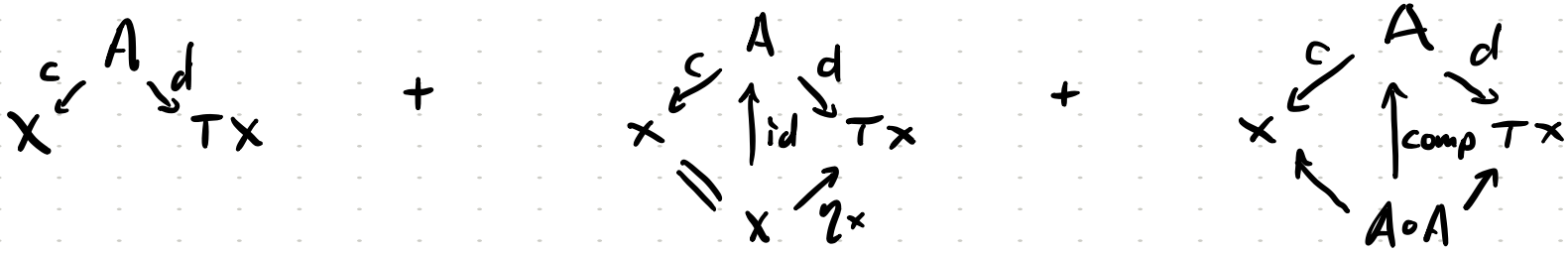
$$X = X \xrightarrow{g} TY \circ Y = Y \xrightarrow{k} TZ$$

... and so on

Recall: A category is a monad in $\mathrm{SPAN}(\mathrm{SET}, \mathrm{Id})$,



Definition: For T a cartesian monad on $\mathbb{C}$ a category with chosen binary pullbacks a *T-category* is a monad in the bicategory $\mathrm{SPAN}(\mathbb{C}, T)$.

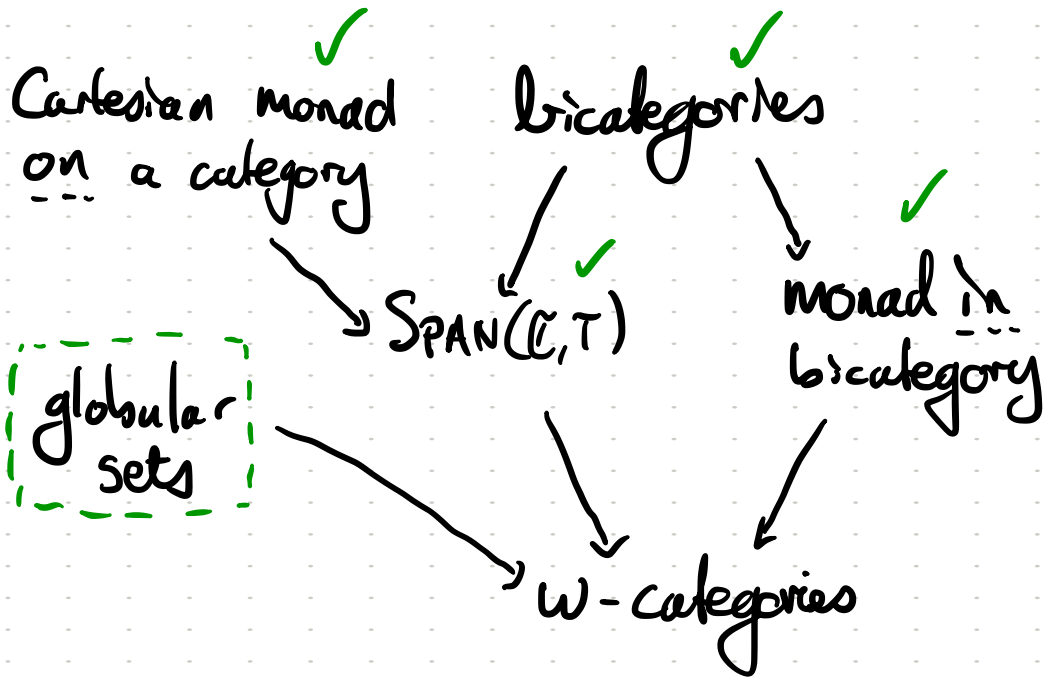


satisfying stuff.

Definition: $(\mathbb{C}, T)\text{-}\mathrm{CAT} :\equiv \mathrm{MND}\left(\mathrm{SPAN}(\mathbb{C}, T)\right)$.

Remark: There is f.f. functor $\mathrm{Alg}(\mathbb{C}, T) \to (\mathbb{C}, T)\text{-}\mathrm{CAT}$ so "many more" examples.

$\mathbb{C}$	T	$(\mathbb{C}, T)$-Cat
Set	Id	categories & functors.
	free monoid	multicategories & multifunctors $a_1, \dots, a_n \xrightarrow{!} b$, $n \in \mathbb{N}$
	VI	
	$1 + (-)$	multicategories with only unary and nullary arrows ... $\rightsquigarrow$ Cat/Set $\simeq$ DiscOpFib
	$M \times (-)$	... Cat/M
$\widehat{\mathbb{G}}_2$ $\{0 \rightrightarrows 1\}$	free category monad	virtual double categories $A \downarrow B$, $A \overset{p}{\nrightarrow} B$, $\begin{array}{ccc} A_0 & \overset{p_1}{\nrightarrow} \cdots \overset{p_n}{\nrightarrow} & A_n \\ f \downarrow & \alpha & \downarrow g \\ B_0 & \xrightarrow[q]{} & B_1 \end{array}$ ⚠


Cartesian monad on a category
bicategories
Span(ℂ,T)
monad in bicategory
globular sets
w-categories

Globular T-categories

Definition: The category of globes has objects $\mathbb{N}$ and

$$\mathbb{G}(m,n) := \begin{cases} 2, & m<n \\ 1, & m=n \\ \emptyset, & m>n \end{cases} \quad \text{or} \quad 0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} 2 \underset{t}{\overset{s}{\rightrightarrows}} \cdots \qquad \begin{array}{l} ts = ss \\ st = tt \end{array}$$

$\widehat{\mathbb{G}} := [\mathbb{G}^{op}, \mathrm{SET}]$ globular[†] sets.

$$X \in \widehat{\mathbb{G}} \begin{cases} X_0 & 0\text{-cells} \ni a, b \\ t \uparrow\uparrow s & \\ X_1 & 1\text{-cells} \ni tf = a \xleftarrow{f} b = sf \\ \vdots & \vdots \qquad\qquad \vdots \end{cases}$$

globular maps $X \xleftarrow{g} Y$

$$\begin{array}{ccc} X_0 & \xleftarrow{g_0} & Y_0 \\ t\uparrow\uparrow s & & t\uparrow\uparrow s \\ X_1 & \xleftarrow{g_1} & Y_1 \\ \vdots & & \vdots \end{array}$$

NB: $\mathbb{N}$ & t reflexive

[†] globe - ular

Fact: The "free strict ω-category monad" on $\hat{\mathbb{G}}$ exists and is Cartesian. Call it T.

$X \in \hat{\mathbb{G}}$, TX is "all free pastings of cells of X"

$a, b, c \in X_0$

$a \overset{f}{\leftarrow} b \quad b \overset{g}{\leftarrow} c \in X_1$

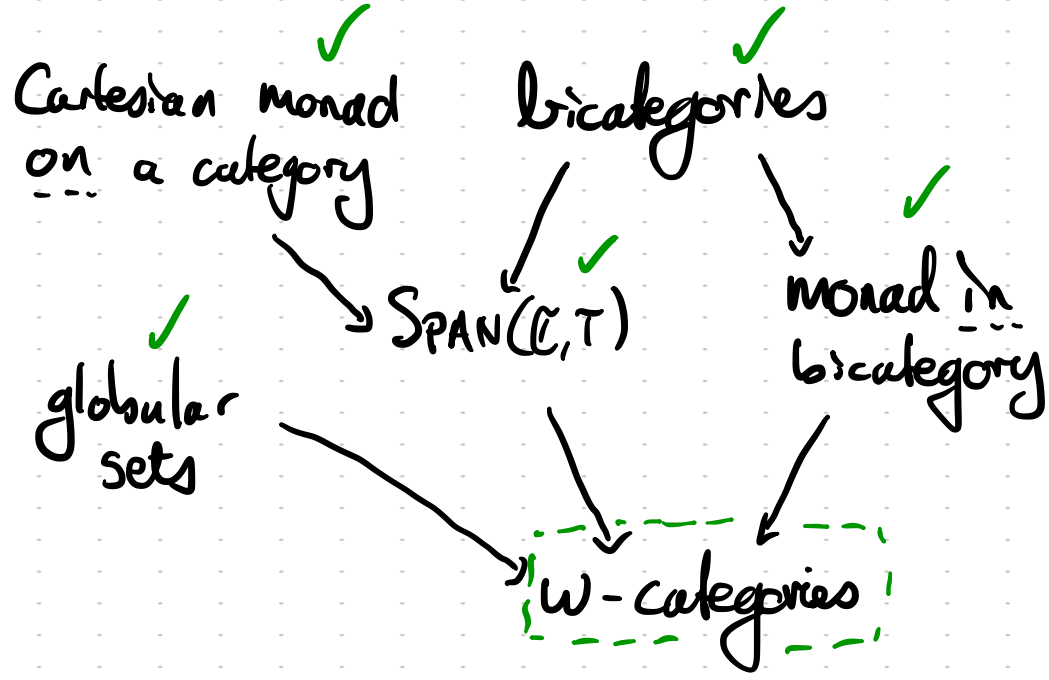
$f \curvearrowright \alpha \curvearrowleft f' \quad (a \to b) \in X_2$

$\Rightarrow \pi \in (TX)_2$

$\pi \equiv a \overset{f}{\leftarrow} b \overset{g}{\leftarrow} c \in (TX)_1$

$\to (TX)_0 \equiv X_0$ **N.B.** Degenerate pastings, part of T

There are things which "look like π" in higher dimensions!


Cartesian monad on a category
bicategories
Span(ℂ,T)
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globular sets
ω-categories

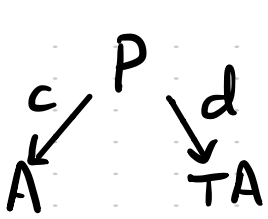
"Definition" (T. Leinster 2001): A proof-relevant ω-category is $(\widehat{\mathbb{G}}, T)$-category where T is the "free strict ω-category monad" equipped with the additional structure of a 'contraction'.

Ok. Blah blah. Here's what it feels like to use this.

$$A \xleftarrow{c} P \xrightarrow{d} TA$$

is the basic data where $P_0 = A_0 \equiv (TA)_0$ and $d_0 = c_0 = id$.

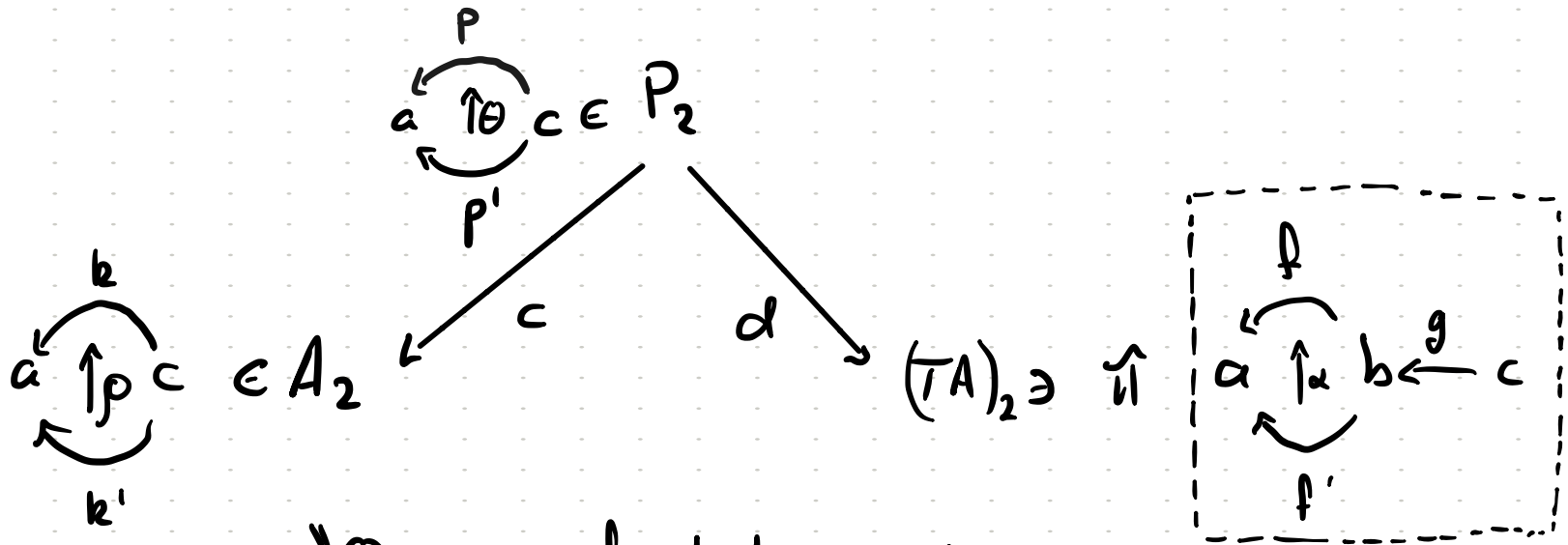
A is the category stuff and P is the proof stuff.



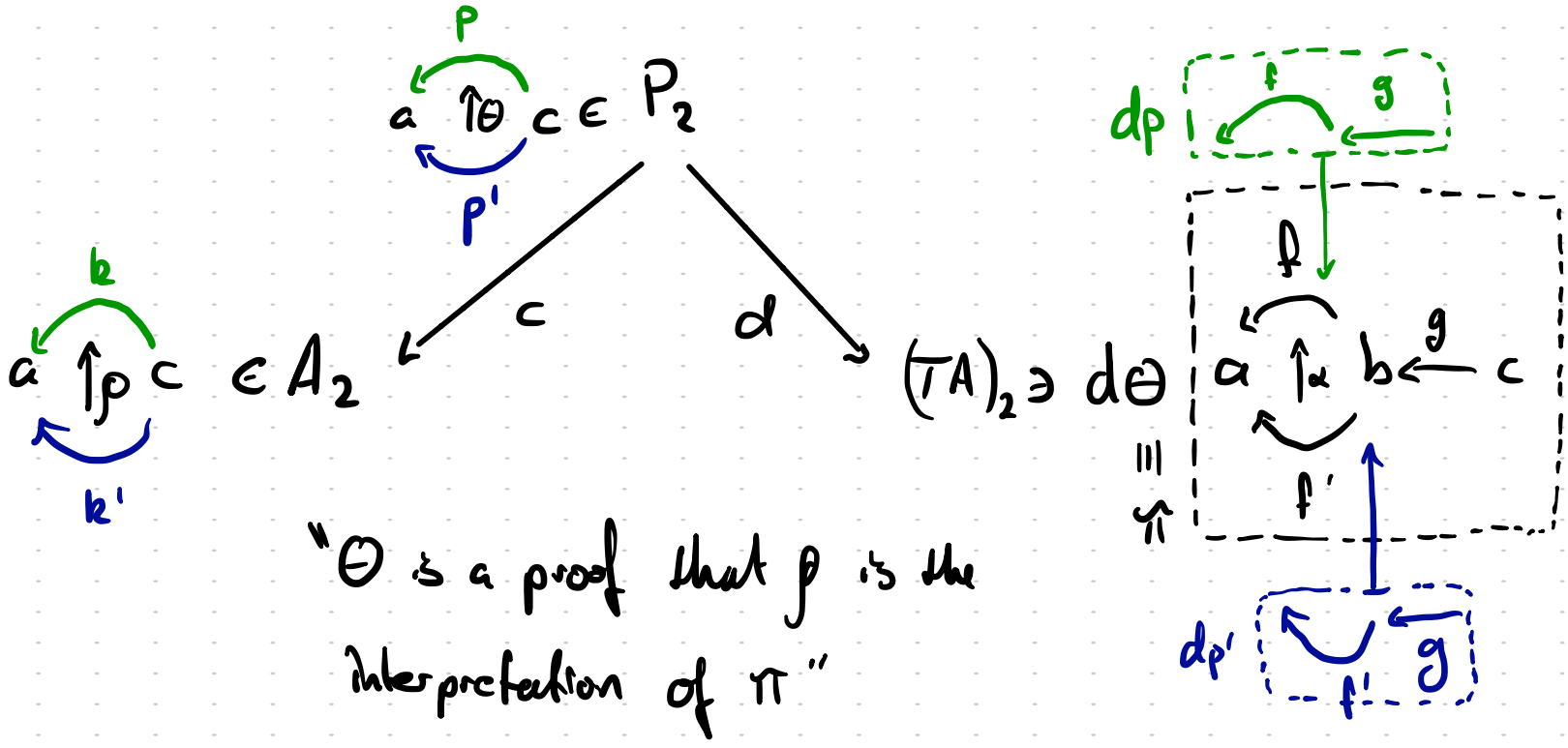
where $P_0 = A_0 \equiv (TA)_0$ and $d_0 = c_0 = id$.

A is the category stuff and P is the proof stuff.

Note: Given some pasting diagram $\pi \in (TA)_n$ in dim. n of A and a cell ρ of dim. n of A we can look at $P(\pi, \rho)$ as the collection of proofs that ρ is an "interpretation" of π.



"Θ is a proof that ρ is the interpretation of π"



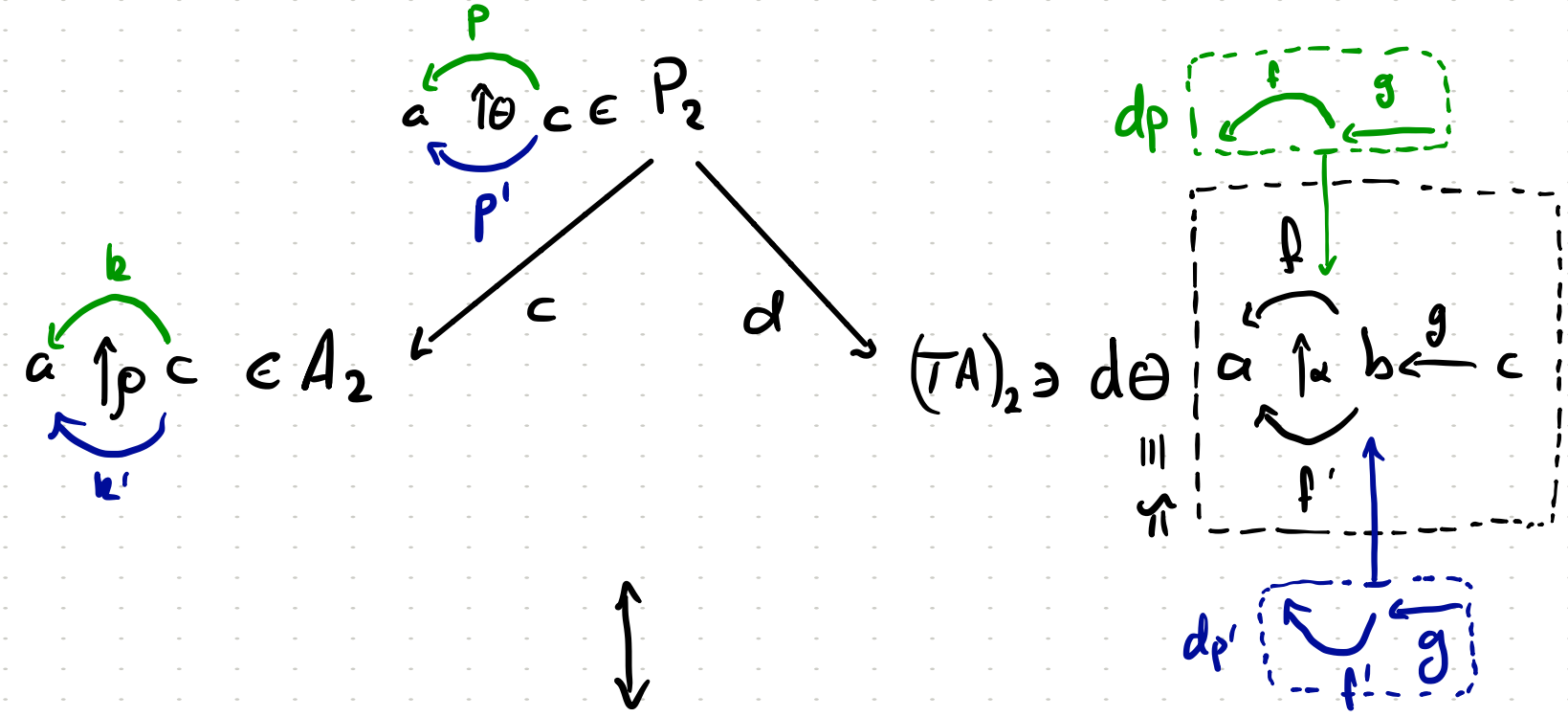
"Θ is a proof that ρ is the interpretation of π"

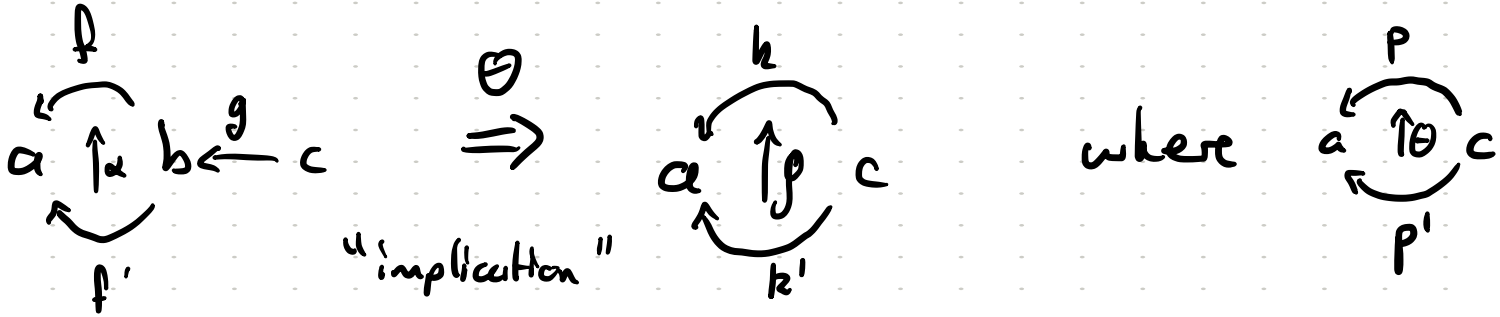
Notice that Θ "lives over" proofs that
- k is the result of $\epsilon\pi$
- k' is the result of $s\pi$

Now we'll examine the other features one by one:

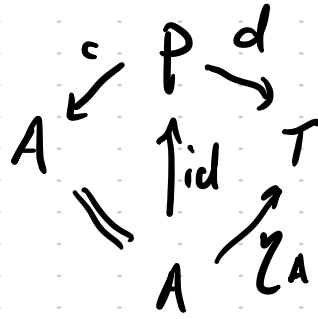
monad-neg , contraction

... but first, notation:





The monad-nerve of $A \xleftarrow{c} P \xrightarrow{d} TA$ (1/2)

The unit 

says that given a cell, say $a \xleftarrow{f} b$ of A_1 we have

$P_1 \ni a \xleftarrow{id_f} b$

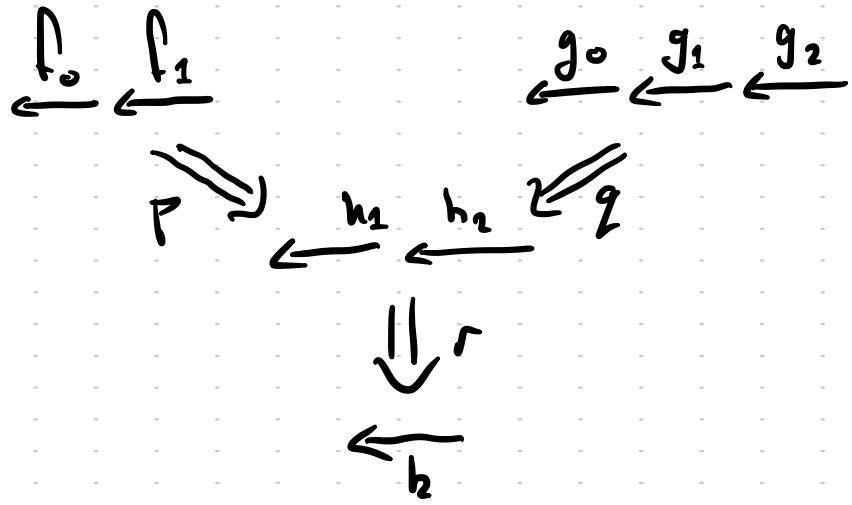
$a \xleftarrow{f} b \in A_1$ $\qquad$ $TA_1 \ni a \xleftarrow{f} b$

$a \xleftarrow{f} b \overset{id_f}{\Longrightarrow} a \xleftarrow{f} b$

"we always have identity proofs"

The monad-ness of $A \xleftarrow{c} P \xrightarrow{d} TA$ (2/2)

The composition of the monad $A \xleftarrow{} P \xrightarrow{} TA$ (with $\mu : P \circ P \to TA$) says that whenever we have proofs arranged as follows



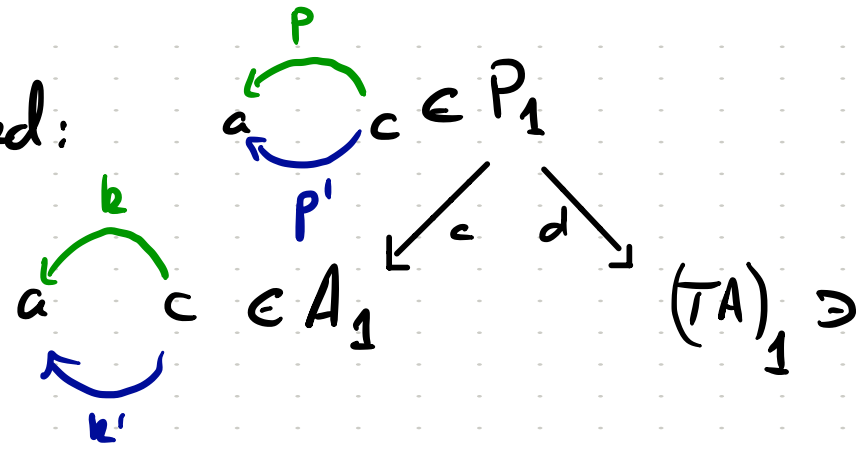
we may perform "proof composition" which gives a proof for the pasting:

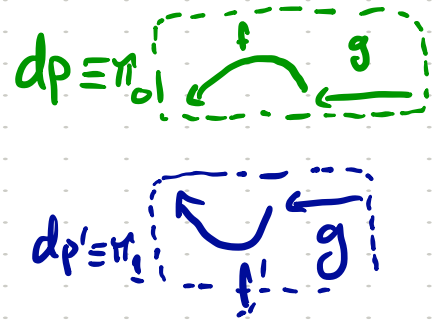
The monadness of $A \xleftarrow{c} P \xrightarrow{d} TA$ (3/2)

Because all of the monad stuff™ is happening essentially vertically in $\mathrm{SPAN}(\hat{\mathbb{G}}, \bar{T})$, proof composition is strictly associative and unital w.r.t. identity proofs.

The last feature: contraction

Suppose we had:





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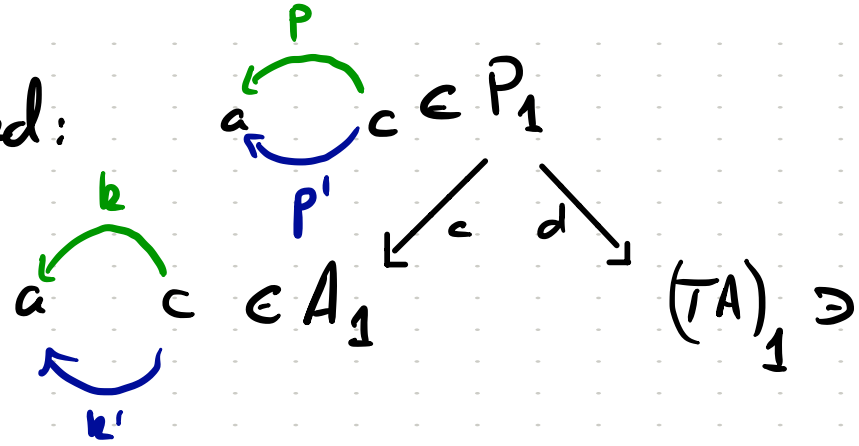
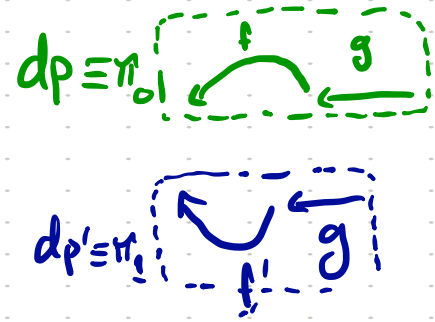
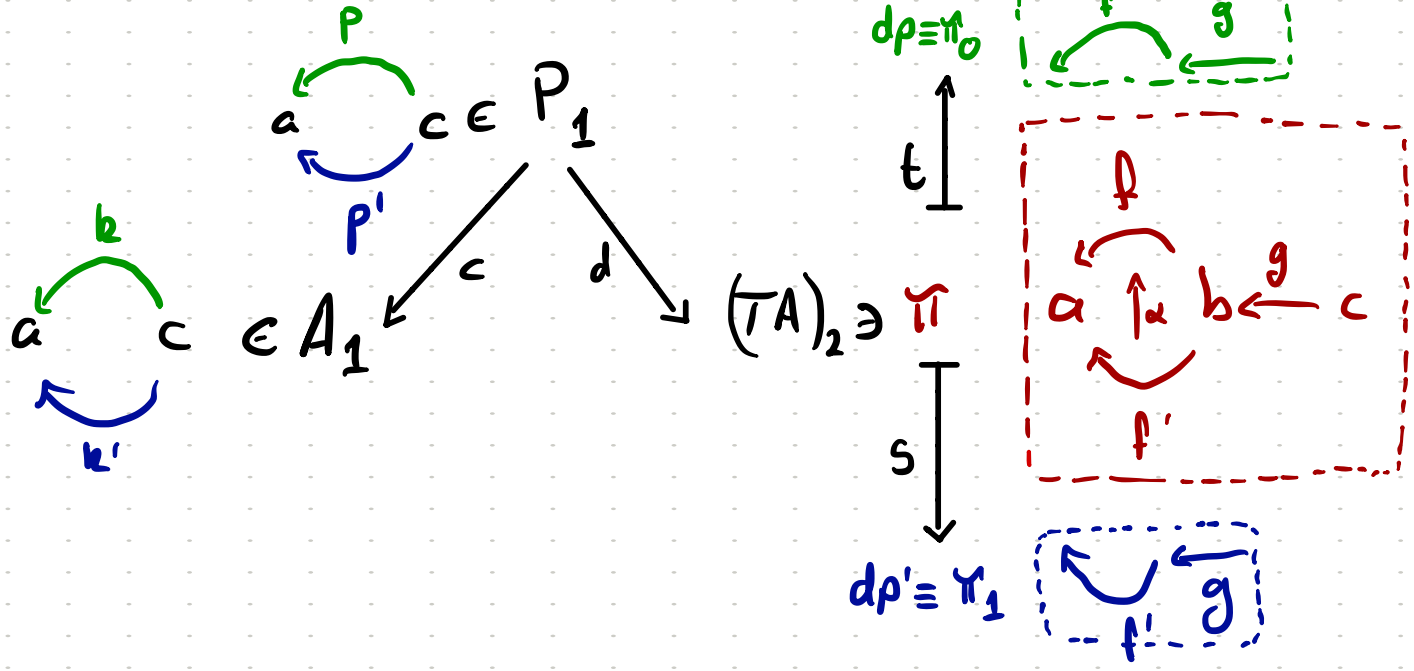




but also $\Pi \in (TA)_2$ with $\begin{cases} s\Pi = \gamma_1 \\ t\Pi = \gamma_0 \end{cases}$



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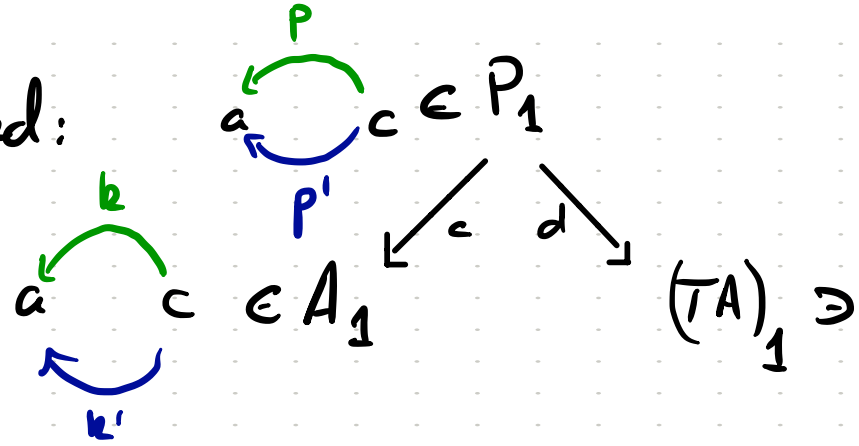

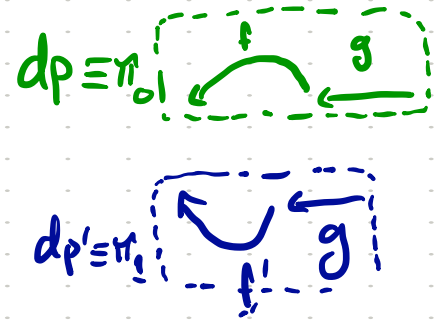

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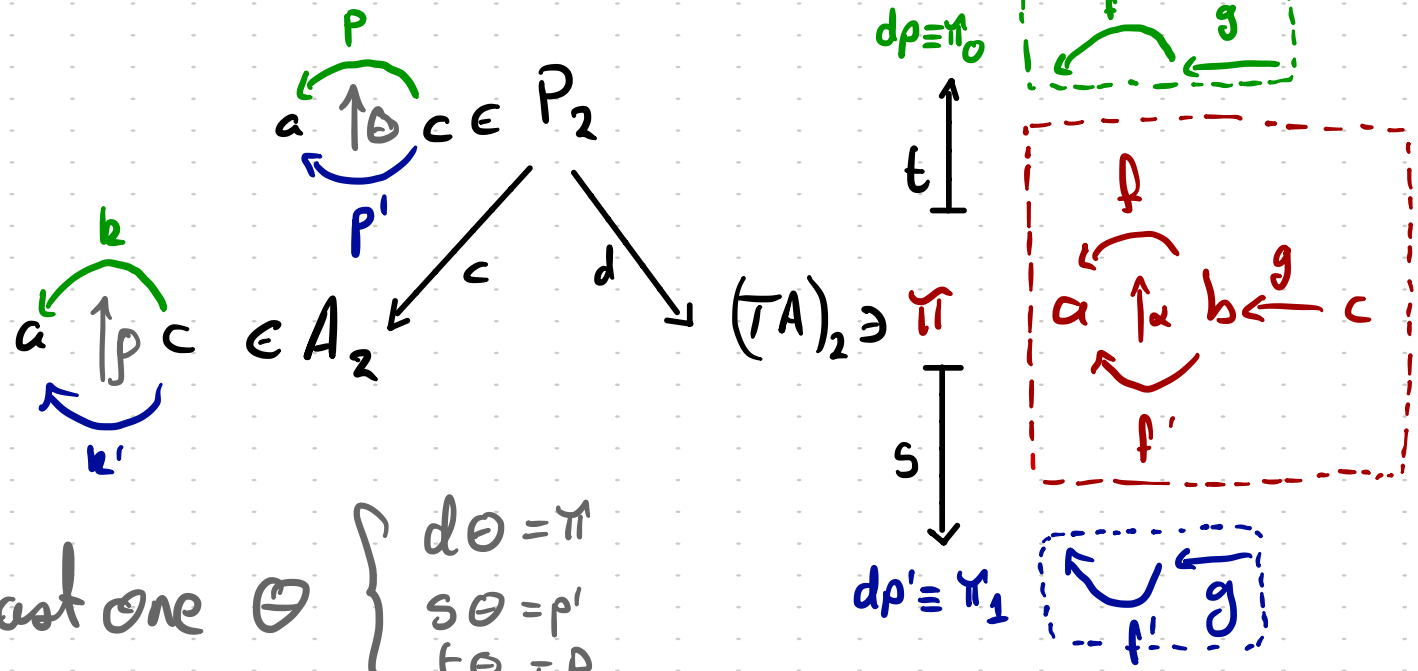
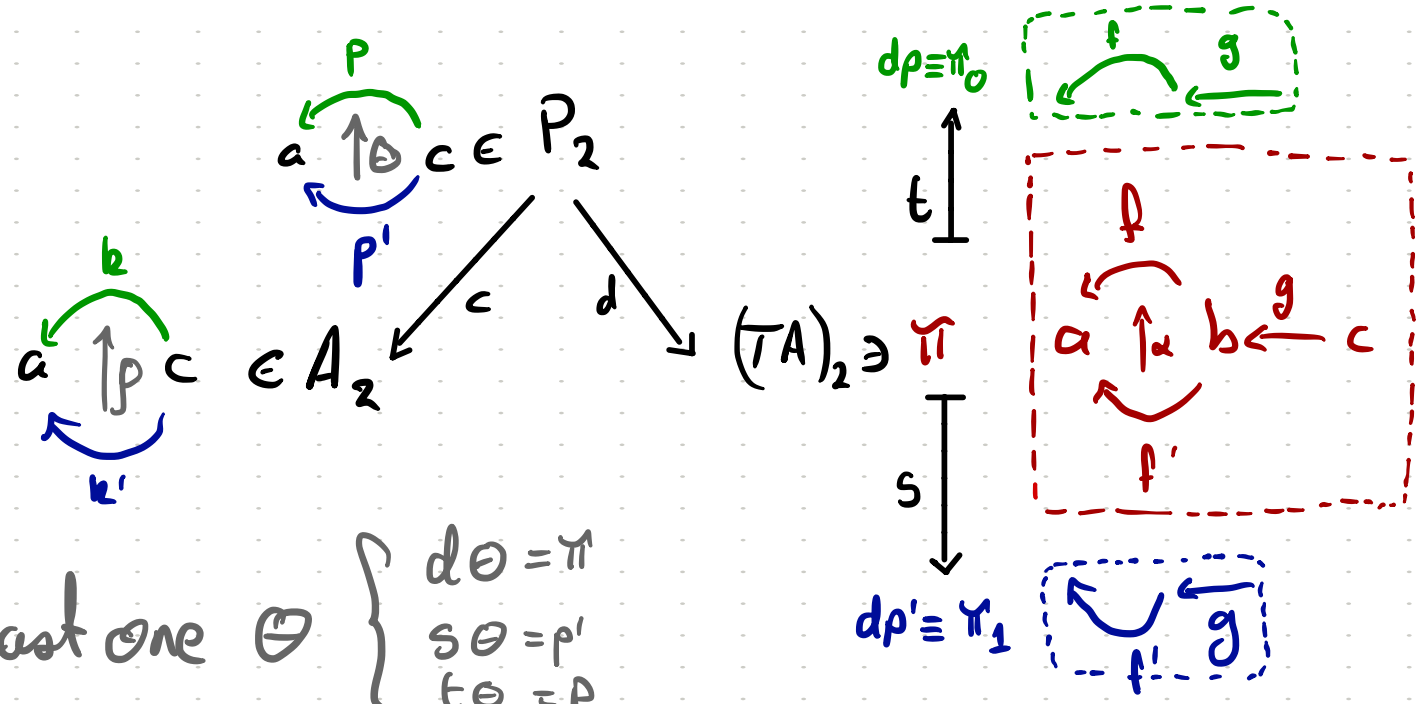


$\rightsquigarrow \exists$ at least one θ $\begin{cases} d\theta = \pi \\ s\theta = p' \\ t\theta = p \end{cases}$



$\rightsquigarrow \exists$ at least one θ $\begin{cases} d\theta = \pi \\ s\theta = p' \\ t\theta = p \end{cases}$

We write $con(p, p', \pi)$ for the θ guaranteed to exist.

"If we have proofs for the interpretation of boundaries of a pasting then there is at least one proof interpretating the pasting between its boundaries"

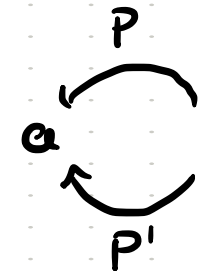
When I said "} at least one" ⇢ ⚠. Let's see what this is like:

$$a \xleftarrow{f} b \xleftarrow{g} c$$

$$P \Downarrow \qquad \Downarrow P'$$

$$a \xleftarrow{h} c \qquad a \xleftarrow{h'} c$$

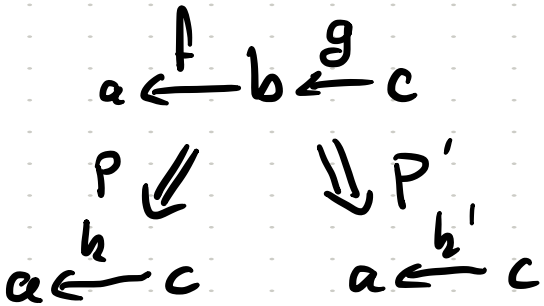
that is

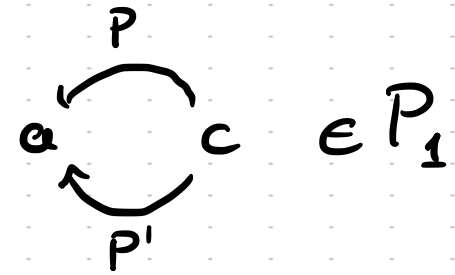


$\in P_1$

Notice • $a \xleftarrow{f} b \xleftarrow{g} c \in (TA)_2$.

When I said "} at least one" ⇢ ⚠. Let's see what this is like:

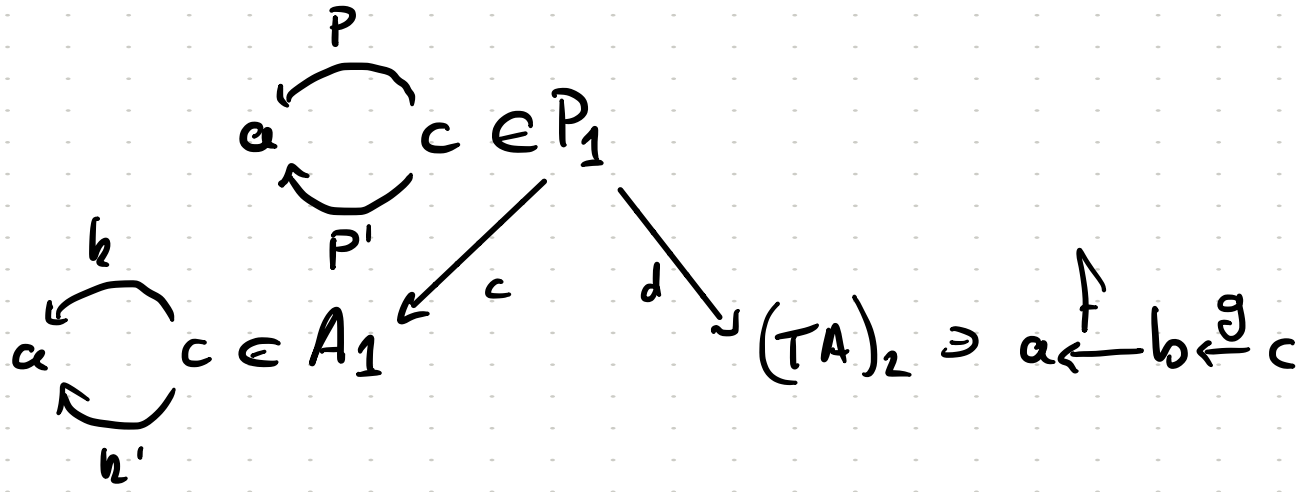


that is 

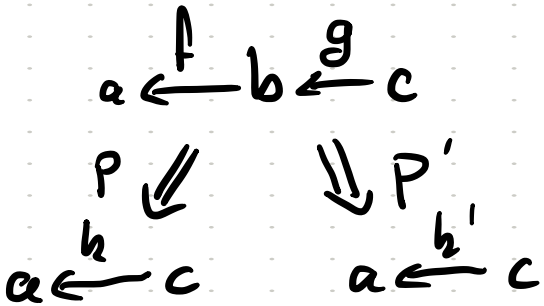
$\in P_1$

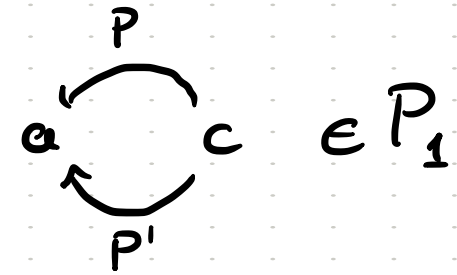
<u>Notice</u> • $a \xleftarrow{f} b \xleftarrow{g} c \in (TA)_2$.

•



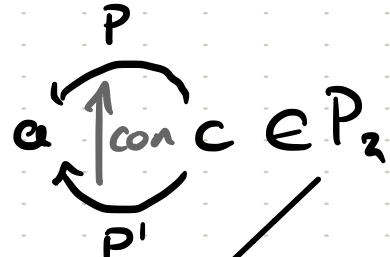
When I said "} at least one" ⟶ ⚠. Let's see what this is like:



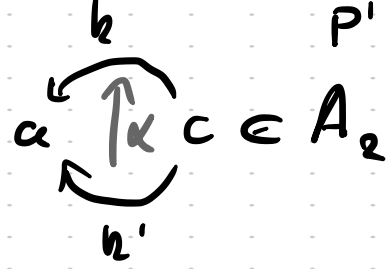
that is 

$\in P_1$

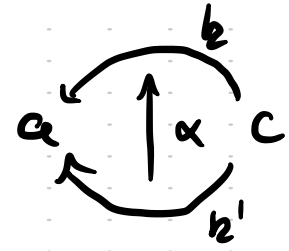
<u>Notice</u> • $a \xleftarrow{f} b \xleftarrow{g} c \in (TA)_2$.

•


$\in P_2$

by contraction


$\in A_2$ (c) (d) $(TA)_2 \ni a \xleftarrow{f} b \xleftarrow{g} c$

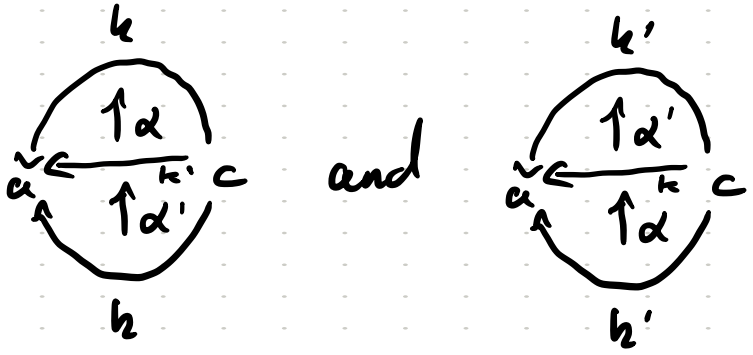
• $a \xleftarrow{f} b \xleftarrow{g} c \overset{con(p,p',\dots)}{\Longrightarrow}$ 

Of course the situation was symmetric,

$$a \xleftarrow{f} b \xleftarrow{g} c$$

$\mathrm{con}(\cdots)$ $\qquad$ $\mathrm{con}(\cdots)$

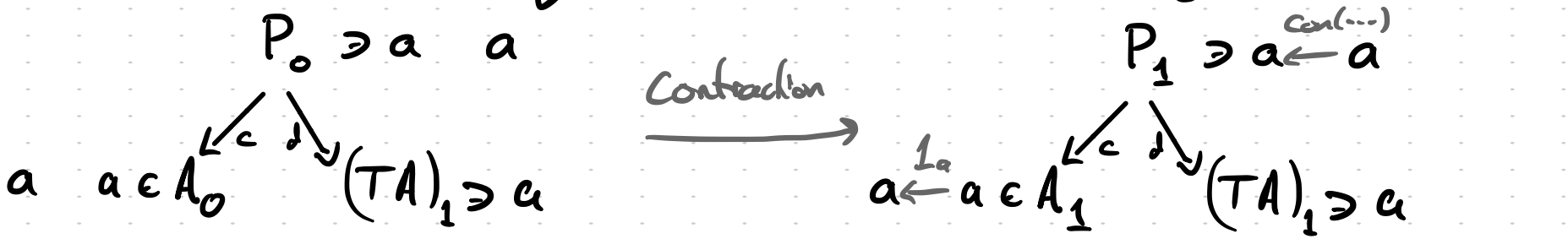
and we may 'compose'



and compare to 'the' identity!

The summary is: 'different' proofs for the same arrangement of cells give interpretations which are 'equivalent'!

Incidentally, what is 'the' identity?



$a \Downarrow_{con(\dots)} a \xleftarrow{1_a} a$ and we can check if $a \xleftarrow{1_a} a \xleftarrow{f} b \xrightarrow{p} a \xleftarrow{b} b$ then "k.f."

(but not nec. only one)

Present results:

1. (Leinster) ω-categories truncate correctly to sets, categories, and enabicategories
2. Products and pullbacks of ω-categories

$\rightsquigarrow$ Definitions of pseudofunctors and K-transfors

Work in progress:

1. $\omega\mathrm{CAT}(\mathbb{B}, \mathbb{A})$ an ω-cat
2. $\mathbb{A}(a, b)$ an ω-cat
3. Import this into HoTT w/ some M and W types.
4. Understand the 'sameness' structure kuf