On the structure of $\mathbf{C}_T$

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**Note:** These are notes on a talk I gave at the graduate student topology seminar at Johns Hopkins in the summer of 2012 (I think) on Chapter 2 of Ravenel’s Orange Book (Nilpotence and Periodicity in Stable Homotopy Theory). If you have any comments or notice any mistakes (of which there are probably many) please email me.

# 1 Review of $\mathbf{MU}$ and formal group laws

Let’s begin by reviewing some things about the connection between $\mathbf{MU}$ and formal group laws.

**-Theorem:** (i) There is a universal formal group law defined over a ring $L$ of the form

$$G(x, y) = \sum_{i,j} a_{i,j} x^i y^j \quad \text{with} \quad a_{i,j} \in L$$

such that for any formal group law $F$ over any ring $R$ there is a unique ring homomorphism $\theta : L \to R$ such that

$$F(x, y) = \sum_{i,j} \theta(a_{i,j}) x^i y^j$$

(ii) (Lazard’s Theorem) $L$ is the polynomial algebra $\mathbb{Z}[x_1, x_2, ...]$.

We proved Lazard’s theorem last semester. We also proved that $\mathbf{MU}_*$ is isomorphic to $\mathbb{Z}[x_1, x_2, ...]$, so $\mathbf{MU}_*$ and $L$ are both polynomial algebras with generators in even degrees (positive even degrees in $\mathbf{MU}_*$, and negative even degrees in $L$).

The usual complex orientation on $\mathbf{MU}$ gives rise to a formal group law, and thus we have a homomorphism $\theta : L \to \mathbf{MU}_*$.

**Quillen’s Theorem:** $\theta$ is an isomorphism. In other words, the formal group law associated with $\mathbf{MU}$ is the universal one.

Given this isomorphism, we can regard $\mathbf{MU}_*(X)$ as an $L$-module.
2 The category $\Gamma$

We begin by defining $\Gamma$, the group of coordinate changes.

**Definition:** Let $\Gamma$ be the group of power series over $\mathbb{Z}$ having the form

$$\gamma = x + b_1 x^2 + b_2 x^3 + \ldots$$

where the group operation is functional composition.

$\Gamma$ acts on the Lazard ring $L$ as follows. Let $G(x, y)$ be the universal formal group law over $L$. Notice that

$$\gamma^{-1}(G(\gamma(x), \gamma(y)))$$

is also a formal group law over $L$. Thus, it must be induced by a homomorphism from $L$ to itself. Since $\gamma$ is an invertible power series, this homomorphism must have an inverse (since we have the formal group law $\gamma(G(\gamma^{-1}(x), \gamma^{-1}(y)))$), so it is an automorphism. This gives the desired action of $\Gamma$ on $L$.

It turns out that $\Gamma$ also acts naturally on $MU_*(X)$.

**Note:** We can describe $\Gamma$ in a different way which explains this action and is useful, both for computations and for proving some of the theorems below. There is a subHopf algebra, $B$, of the Hopf algebroid $MU_*MU$. Since our Hopf algebra is a cogroup object in the category of $\mathbb{Z}/p$-algebras, the set $\text{Hom}_{\mathbb{Z}/p}(B, \mathbb{Z})$ is a group which turns out to be $\Gamma$. This is covered in appendix B and would be a good subject for a talk later this summer.

This action is compatible with the action of $\Gamma$ on $MU_*(pt)$ defined above. That is, if $x \in MU_*(X), \lambda \in L = MU_*(pt)$, and $\gamma \in \Gamma$, then

$$\gamma(\lambda x) = \gamma(\lambda)\gamma(x)$$

Also, the action of $\Gamma$ commutes with homomorphisms induced by continuous maps.

**Definition:** Let $C\Gamma$ denote the category of finitely presented graded $L$-modules equipped with an action of $\Gamma$ compatible with the action of $\Gamma$ on $L$ as above. Let $FH$ denote the category of finite CW-complexes and homotopy classes of maps between them.

Thus, $MU_*$ is a functor from $FH$ to $C\Gamma$. The nilpotence, periodicity, and chromatic convergence theorems show that the category $C\Gamma$ captures a lot of information about the category of $FH$. Much of this talk can be thought of as an algebraic version of Jon’s talk two weeks ago.

**Definition:** For each integer $n$, the $n$-series $[n](x)$ is given by

$$[1](x) = x$$

$$[n](x) = F(x, [n-1](x)) \quad \text{for } n > 1$$

$$[-n](x) = i([n](x))$$
We are particularly interested in the $p$-series. In characteristic $p$, the $p$-series of any formal group law always has leading term $ax^q$ where $q = p^h$ for some integer $h$.

**Definition:** Let $F(x, y)$ be a formal group law over a ring in which the prime $p$ is not a unit. If the mod $p$ reduction of $[p](x)$ has the form

$$[p](x) = ax^{p^h} + \text{higher terms}$$

with $a$ invertible, then we say that $F$ has height $h$ at $p$. If $[p](x) \cong 0 \mod p$, then the height is infinity. There is a nice classification theorem for formal group laws over the algebraic closure of $F_p$.

**Theorem:** Two formal group laws over the algebraic closure of $F_p$ are isomorphic if and only if they have the same height.

Let $v_n$ denote the coefficient of $x^{p^n}$ in the $p$-series for the universal formal group law. It can be shown that $v_n$ is an indecomposable element in $L$. Let $I_{p,n}$ denote the prime ideal $(p, v_1, ..., v_n-1)$.

**Invariant prime ideal theorem:** The only prime ideals in $L$ which are invariant under the action of $\Gamma$ are the $I_{p,n}$ (including $n = \infty$). Moreover, in $L/I_{p,n}$ for $n > 0$, the subgroup fixed by $\Gamma$ is $\mathbb{Z}/p[v_n]$. In $L$ itself, the invariant subgroup is $\mathbb{Z}$.

This considerably simplifies the structure of $L$ that we care about. $L$ has many prime ideals, but the only ones we care about are the $I_{p,n}$. Furthermore, we have the following very useful theorem on filtrations of modules in $C\Gamma$.

**Landweber filtration theorem:** Every module $M$ in $C\Gamma$ admits a finite filtration

$$F_1 M \subset F_2 M \subset ... \subset F_k = M$$

such that each subquotient is isomorphic to a suspension of $L/I_{p,n}$ for some prime $p$ and some finite $n$.

We now have the following corollary of the Landweber filtration theorem, which leads to the algebraic notion of the type of a CW complex which we discussed in Jon’s talk.

**Corollary:** Suppose $M$ is a $p$-local module in $C\Gamma$ and $x \in M$.

(i) If $v_n^{-1}M = 0$, then $v_{n-1}^{-1}M = 0$.

(ii) If $M$ is nontrivial, then so is $v_n^{-1}M$ for sufficiently large $n$.

(iii) If $v_n^{-1}M = 0$, then there is a positive integral $k$ such that multiplication by $v_n^k$ commutes with the action of $\Gamma$.

(iv) Conversely, if $v_n^{-1}M$ is nontrivial, then there is no positive integer $k$ such that multiplication by $v_n^k$ in $M$ commutes with the action of $\Gamma$ on $M$.

**Note:** The first statement should be compared to the fact about $K(n)$ that if $\overline{K(n)}_*(X) = 0$, then $\overline{K(n-1)}_*(X) = 0$. The second statement is analogous to the fact that if $X$ is not
contractible, then $\overline{K(n)}_*(X)$ is nontrivial for $n$ large. In fact, the functor $v^{-1}_nMU_*(X)_{(p)}$ is a homology theory which vanishes on a finite $p$-local CW-complex $X$ if and only $\overline{K(n)}_*$ does. So we could replace $\overline{K(n)}_*$ with $v^{-1}_nMU_*$ in the statement of the periodicity theorem.

The third statement is an algebraic analog of the periodicity theorem. If we think of the action of $\Gamma$ as the action of $MU_*MU$ by homology operations, then (iii) says that our algebraic $v_n$-map should commute with all stable homology operations. This will certainly be true if the algebraic $v_n$-map is induced by a continuous map $\Sigma^dX \to X$ by applying $MU_*$.  

**Proof:**  
(i) Suppose that $v^{-1}_nM = 0$. Let $F_1M \subset F_2M \subset \ldots \subset F_kM$ be the Landweber filtration of $M$. We claim that each Landweber subquotient is of the form $L/I_{p,m}$ with $m \geq n$. Suppose otherwise. That is, suppose there exists some subquotient $F_iM/F_{i-1}M$ isomorphic to $L/I_{p,m}$ with $m < n$. Let $x \in M$ be such that $x \in F_iM$ but $x \notin F_{i-1}M$. Since $v^{-1}_nM = 0$, there exists some $l$ such that $v^l_nx = 0$. However, since $v^l_n \notin I_{p,m}$ when $m < n$, we have that $v^l_nx \neq 0$ in the Landweber subquotient, which is a contradiction. Thus, we must have that $m \geq n$ in each Landweber subquotient. This proves the claim.

Now consider $v^l_{n-1}x$ for some $l$ and $x \in M$. Since every subquotient has the form $L/I_{p,m}$ with $m \geq n$, we have multiplication by $v_{n-1}$ is zero on each subquotient. This means that $v^l_{n-1}x$ and $v^{l+1}_{n-1}x$ cannot both be nonzero in the same subquotient since one is $v_{n-1}$ times the other, hence 0. But the Landweber filtration is finite, so there are only finitely many subquotients. Thus, there is a smallest $l$ such that $v^l_{n-1}x$ is zero in every subquotient and so is zero in $M$ (the largest this $l$ could be is the length of the filtration minus one). Thus, $v^{-1}_nM = 0$.

(ii) Choose $n$ large so that each Landweber subquotient has the form $L/I_{p,m}$ for $m < n$. Then for any $x \in M$, if we identify $x$ with its image in the Landweber subquotient in which it is nonzero, we may multiply it by any power of $v_n$ and it will not be zero since $v_n \notin I_{p,m}$. Thus, $x$ is not $v_n$-torsion in $M$. So $v^{-1}_nM \neq 0$.

(iii) Suppose $v^{-1}_nM = 0$. Let $k$ be the length of the Landweber filtration. We claim that $M$ is annihilated by $I_{p,n}$. First, notice that $v^{-1}_iM = 0$ for $i \leq n - 1$ by (i). The image of any $x \in M$ in any Landweber subquotient is annihilated by $I_{p,n}$. Let $g = p^{j_0}v^1_1 \cdots v^{i_{n-1}}_n$ be an arbitrary degree $j$ monomial on the generators of $I_{p,n}$. We can find a chain of divisors $g_1 = 1|g_2| \ldots |g_j|g$ where $g_i = h_{g_{i-1}}$ for $h \in I_{p,n}$. (For example, if $g = v^2_1v_2$, we have $1|v_1^2|v_1^2v_2 = g$. Since any two elements in the chain differ by a factor in $I_{p,n}$ and multiplication by an element in $I_{p,n}$ is zero in any subquotient, we must have that the product of each divisor with $x$ in the chain belongs to a distinct subquotient. But there are $j + 1$ elements in the chain (including $g$) but only $j$ subquotients, which forces $gx$ to be zero. This proves the claim.

Now we claim that multiplication by $v^{p^j}_n$ commutes with the action of $\Gamma$. Let $x \in M$. We need to show that for any $\gamma \in \Gamma$, 

$$\gamma(v^p_nx) = v^p_n\gamma(x)$$
Since $M \in C\Gamma$, we have
\[ \gamma(v^p_n x) = \gamma(v^p_n)\gamma(x) \]
We claim that $\gamma(v^p_n) = v^p_n$ modulo $I^2_{p,n}$. First,
\[ \gamma(v^p_n) = \gamma(v_n)^p \]
By the invariant prime ideal theorem, we have that $\mathbb{Z}[v_n]$ is $\Gamma$-invariant in $L/I_{p,n}$. That is,
\[ \gamma(v_n) = v_n + e, \quad e \in I_{p,n} \]
Then
\[ \gamma(v_n)^p = (v_n + e)^p = v^p_n + e' \]
where $e' \in I^2_{p,n}$ (use the binomial theorem and don’t forget to look at the binomial coefficients, since $p \in I_{p,n}$ counts). Thus,
\[ \gamma(v^p_n x) = (v^p_n + e')x = v^p_n x \]
Since $e'x = 0$ as we proved in our claim above. Thus, multiplication by $v^p_n$ commutes with the action of $\Gamma$ when $v^{-1}_n M = 0$.

(iv) Suppose there exists a $k$ such that multiplication by $v^k_n$ commutes with the action of $\Gamma$ on $M$. Then multiplication by $v^k_n$ must also be equivariant on each Landweber subquotient. We claim that each Landweber subquotient must have the form (suspension of) $L/I_{p,m}$ for $m \geq n$. Indeed, we have in any subquotient that $\gamma(v^k_n) = v^k_n$, so $v^k_n$ is invariant under the action of $\Gamma$. But by the invariant prime ideal theorem, the only $\Gamma$ invariant elements in $L/I_{p,m}$ are in $\mathbb{Z}[v_m]$. So multiplication by $v^k_n$ cannot be $\Gamma$ equivariant unless each subquotient has $m \geq n$. But by the same argument as the one given in (i), this implies that every element in $M$ is $v_{n-1}$ torsion, i.e. $v^{-1}_{n-1} M = 0$.

**Definition:** A $p$-local module $M$ in $C\Gamma$ has type $n$ if $n$ is the smallest integer with $v^{-1}_n M$ nontrivial. A homomorphism $f: \Sigma^d M \to M$ is a $v_n$-map if it induces an isomorphism in $v^{-1}_n M$ and the trivial homomorphism in $v^{-1}_m M$ for $m \neq n$.

**Corollary:** If $M \in C\Gamma$ is a $p$-local module with $v^{-1}_{n-1} M$ nontrivial, then $M$ does not admit a $v_n$-map.

**Proof:** Suppose that $M \in v^{-1}_{n-1} M \neq 0$. Then $M$ has type $m$ for some $m < n$. Then $v^{-1}_m M \neq 0$ and $v^{-1}_{n-1} M \neq 0$. Thus, $v^{-1}_m v^{-1}_{n-1} M = \neq 0$. On the other hand, if $f$ is a $v_n$-map, then $v^{-1}_n M$ is an isomorphism and $v^{-1}_m f = 0$. But since localization is exact, we must have that $v^{-1}_n v^{-1}_m f$ is both an isomorphism and zero, which is impossible since $v^{-1}_m v^{-1}_{n-1} M \neq 0$.

### 3 Thick subcategories

**Definition:** A full subcategory $C$ of $C\Gamma$ is thick if it satisfies the following axiom:
If
\[0 \to M' \to M \to M'' \to 0\]
is a short exact sequence in \(\mathcal{C}\Gamma\), then \(M\) is in \(\mathcal{C}\) if and only if \(M'\) and \(M''\) are.

A full subcategory \(\mathcal{F}\) of \(\mathcal{FH}\) is thick if it satisfies the following two axioms:

(i) If
\[X \to Y \to C_f\]
is a cofibration in which two of the three spaces are in \(\mathcal{F}\), then so is the third.

(ii) If \(X \lor Y\) is in \(\mathcal{F}\) then so are \(X\) and \(Y\).

Using the Landweber filtration theorem, we can classify the thick subcategories of \(\mathcal{C}\Gamma(p)\):

**Theorem:** Let \(\mathcal{C}\) be a thick subcategory of \(\mathcal{C}\Gamma(p)\) (\(p\)-local modules in \(\mathcal{C}\Gamma\)). Then \(\mathcal{C}\) is either all of \(\mathcal{C}\Gamma(p)\), the trivial subcategory, or consists of all \(p\)-local modules \(M\) in \(\mathcal{C}\Gamma\) with \(v_{n-1}M = 0\). We denote such a category by \(\mathcal{C}_{p,n}\).

First, note that given \(M\) and \(N\) in \(\mathcal{C}\Gamma\), we can define \(\Gamma\)-actions on \(M \otimes_{MU} N\) and \(\text{Hom}(M,N)\) by
\[\gamma(m \otimes n) = \gamma(m) \otimes \gamma(n)\]
and
\[\gamma(f)(m) = \gamma(f(\gamma^{-1}(m)))\]
(Note: the homomorphisms in \(\text{Hom}(M,N)\) are not required to be \(\Gamma\)-equivariant.)

Before we prove the theorem, we need a proposition.

**Proposition:** If \(\mathcal{C} \subset \mathcal{C}\Gamma\) is thick and \(M\) is in \(\mathcal{C}\), then so are \(N \otimes M\) and \(\text{Hom}(N,M)\) for any \(N \in \mathcal{C}\Gamma\).

**Proof:** Since \(N \in \mathcal{C}\Gamma\) is finitely presented as an \(L\)-module, we may use the Landweber filtration theorem to prove that it has a finite free resolution in \(\mathcal{C}\Gamma\):
\[0 \leftarrow N \leftarrow F_0 \leftarrow F_1 \leftarrow ... \leftarrow F_n \leftarrow 0\]
(I'm not sure how we get a finite resolution using the Landweber filtration theorem. I haven't thought about it very much, but if anyone has any thoughts, please let me know.)

If \(M\) is in \(\mathcal{C}\), then \(F_i \otimes M\), a direct sum of copies of \(M\), is also in \(\mathcal{C}\) (in general, if \(M_1\) and \(M_2\) are in \(\mathcal{C}\), then the short exact sequence \(0 \to M_1 \to M_1 \oplus M_2 \to M_2\) shows that \(M_1 \oplus M_2\) is in \(\mathcal{C}\). Also being a direct sum of copies of \(M\), \(\text{Hom}(F_i,M)\) must also be in \(\mathcal{C}\). Ravenel claims that by induction on \(n\) (the length of the resolution), we can show that \(M \otimes N\) and \(\text{Hom}(N,M)\) are both in \(\mathcal{C}\). I am having trouble showing this because a thick subcategory of \(\mathcal{C}\Gamma\) is closed under subobjects, quotients, and extensions from \(\mathcal{C}\Gamma\). In general, if I have a module in \(\mathcal{C}\Gamma\), I don't know if one of its submodules is \(\mathcal{C}\Gamma\). So even though we have that \(M \otimes N\) is a quotient of \(F_i \otimes M\), and \(F_i \otimes M\) is in \(\mathcal{C}\), we only get that \(M \otimes N\) is in \(\mathcal{C}\) if the kernel of \(F_i \otimes M \to M \otimes N\) is in \(\mathcal{C}\Gamma\)...(again, please let me know if you have an idea).
Now we are ready to prove our “algebraic” thick subcategory theorem:

**Proof:** Suppose that $C \subset C\Gamma(p)$ is thick. Choose the smallest $n$ so that $C_{p,n} \supset C$. We want to show that $C \supset C_{p,n}$. Let $N$ be in $C_{p,n}$ and let $M$ be any module in $C$ but not in $C_{p,n+1}$ (so $M$ has type $n$). $M$ will be an accessory to showing that $N$ is in $C$.

Any module in $C_{p,n}$ is annihilated by some power of $I_{p,n}$, as we showed in the proof of part (iii) of the corollary above (the power of $I_{p,n}$ needs to be no greater than the length of the Landweber filtration of the module). Thus, we have that $N$ is annihilated by $I_{k}^{k}$ for some $k$. The natural map $s : MU_{(p)*} \rightarrow End(N)$ has as its kernel the annihilator ideal of $N$, denoted by $Ann(N)$. One of Landweber’s papers shows that it is $\Gamma$-invariant. Thus, we have that $I_{k}^{k} \subseteq Ann(N)$

Now we turn to $Ann(M)$. Since $v_{n}^{-1}M$ is nontrivial, for $m \geq n$ no power of $v_{m}$ can be in $Ann(M)$ (by part (i) of the corollary above). $Ann(M)$ must be contained in some $\Gamma$-invariant prime ideal (Why??), and thus, this ideal must be $I_{p,n}$. Thus,

$$Ann(M)^{k} \subseteq I_{p,n}^{k} \subseteq Ann(N)$$

Now, consider the exact sequence

$$0 \rightarrow Ann(M)^{k} \rightarrow MU_{(p)*} \rightarrow End^{\otimes k}$$

where the right hand map is $s^{\otimes k}$. (Why is $ker(s^{\otimes k})$ contained in the image of the left hand map??). Tensoring with $N$ we have

$$0 \rightarrow N \otimes Ann(M)^{k} \rightarrow N \otimes MU_{(p)*} \rightarrow N \otimes End(M)^{\otimes k}$$

Since $Ann(M)^{k} \subseteq Ann(N)$, we have that $N \otimes Ann(M)^{k} = 0$, so the exact sequence reduces to

$$0 \rightarrow N \rightarrow End(M)^{\otimes k} \otimes N$$

Since $N \in C\Gamma$ and $M \in C$, our previous proposition implies that $N \in C$. Thus, $C_{p,n} \subseteq C$, so $C = C_{p,n}$. □

Motivated by the results in the algebraic category, we would hope that there is an analogous result about thick subcategories of $FH_{(p)}$. This thick subcategory theorem will follow from the nilpotence theorem and will be a major ingredient in the proof of the periodicity theorem (we give a brief explanation below).

**Thick subcategory theorem:** Let $F$ be a thick subcategory of $FH_{(p)}$, the category of $p$-local finite CW-complexes. Then $F$ is either all of $FH_{(p)}$, the trivial subcategory, or consists of all $p$-local finite CW-complexes $X$ with $v_{n-1}^{-1}MU_{*}(X) = 0$. We denote such a category by $F_{p,n}$.

So there are two nested sequences of thick subcategories

$$FH_{(p)} = F_{p,0} \supset F_{p,1} \supset F_{p,2} \supset \ldots \supset \{pt\}$$

and

$$CT = C_{p,0} \supset C_{p,1} \supset C_{p,2} \supset \ldots \supset \{0\}$$
and the functor $MU_*(-)$ sends one nested subsequence to the other. The category $CT$ is much easier to study than $FH_{(p)}$. For example, it was not known until 1983 whether the $F_{p,n}$ were even nontrivial for all but some small values of $n$. This was an argument given by Mitchell. I will write up a sketch of it sometime.

We conclude with the relationship between the thick subcategory theorem and the periodicity theorem. We will show (in Chapter 6) that the collection of complexes admitting periodic self-maps for given $p$ and $n$ forms a thick subcategory (call such a subcategory $D_{p,n}$). The hard part of the proof of the periodicity theorem is to show that these subcategories are nonempty, that is, that there exists just one nontrivial example of a complex of type $n$ with a periodic self-map. Since all complexes in $D_{p,n}$ are of type $n$, we have

$$F_{p,n+1} \subset D_{p,n} \subseteq F_{p,n}$$

Since $D_{p,n}$ is thick, the thick subcategory theorem implies that

$$D_{p,n} = F_{p,n}$$

Now let $X$ be any $p$-local finite CW-complex. Then the thick subcategory generated by $X$ must be $F_{p,n} = D_{p,n}$ for some $n$. So $X$ admits a periodic self-map.