Connective covers of Real Johnson-Wilson theories

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Outline

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Introduction: detecting homotopy classes

- What classes in $\pi_* S^0$ can we detect in the image of $\pi_* S^0 \to \pi_* E$ (the Hurewicz map) for various cohomology theories E?
- The image in $\pi_* KO$ contains η and η^2. In fact, all of the torsion in $\pi_* KO$ is in the Hurewicz image.
- The image in $\pi_* TMF\langle 3 \rangle$ (topological modular forms with level 3 structure) contains (at least) η, ν, and $\overline{\kappa}$. (Mahowald, Rezk)
- What is the Hurewicz image in other cohomology theories? Is there a family of cohomology theories which detects more and more of $\pi_* S^0$?

Chromatic homotopy theory

- Let X be a (p-local, finite) spectrum. There is a filtration on X such that the nth slice is $L_{K(n)}X$, the Bousfield localization of X at Morava K-theory, $K(n)$.
- **Theorem:** (Devinatz and Hopkins)

$$L_{K(n)}S^0 = E_n^{hG(n)}$$

where E_n denotes Morava E-theory, and $G(n)$ denotes the Morava stabilizer group.

- Interesting question: if G is a *finite* subgroup of $G(n)$, what does E_n^{hG} 'see'?
- Results in this vein: resolutions of the $K(2)$-local sphere (Goerss, Henn, Mahowald, Rezk), Kervaire invariant one (Hill, Hopkins, Ravenel)

Chromatic homotopy theory

- Let $E(n)$ denote Johnson-Wilson theory, where $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n][v_n^{-1}]$
- One can show

$$L_{K(n)}E(n) = E_n^{h(\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \ltimes \mathbb{F}_{2^n}^\times)}$$

- $\pi_* L_{K(n)}E(n)$ is torsion-free, so the Hurewicz image is trivial outside of degree 0.
- But there is a C_2-action on $E(n)$ (coming from the C_2-action by complex conjugation on MU) such that $\pi_* L_{K(n)}E(n)^{hC_2}$ contains torsion.

Chromatic homotopy theory

- We would like to compute the Hurewicz image in $\pi_* L_{K(n)} E(n)^{hC_2}$, or better yet $\pi_* E(n)^{hC_2}$.
- Morally, $L_{K(n)} E(n)^{hC_2}$ is $\left(E_n^{h(\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \ltimes \mathbb{F}_{2^n}^{\times})} \right)^{hC_2}$
- Philosophy: control the size of the subgroup of $G(n)$ but work at *all* chromatic heights.
- First, we need to get precise about '$E(n)^{hC_2}$'.

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Defining Real cobordism

There is a fiberwise C_2-action on the canonical n-plane bundle γ_n over $BU(n)$. This extends to an action on its Thom space, $\mathrm{Th}(\gamma_n)$.

Definition

Let α denote the sign representation of C_2. Define

$$M\mathbb{R}(n(1+\alpha)) = Th(\gamma_n)$$

Since $\{n(1+\alpha)\}$ is cofinal among all representations of C_2, we may spectrify to obtain a genuine C_2-equivariant spectrum, $M\mathbb{R}$.

Note: $M\mathbb{R}_\star$ is not actually the cobordism ring of Real manifolds.

Facts about $M\mathbb{R}$ (Hu and Kriz)

- $M\mathbb{R}$ is an E_∞-ring spectrum.
- $M\mathbb{R}$ is 'Real-oriented'. So it supports a formal group law.
- The forgetful map $M\mathbb{R}_\star \to MU_*$ is split by a map of rings $MU_* \to M\mathbb{R}_\star$ classifying the formal group law, where the image of x_i is in degree $i(1+\alpha)$
- There is a Real analog of the Quillen idempotent which produces a C_2-equivariant spectrum $BP\mathbb{R}$ such that $M\mathbb{R}_{(2)}$ splits as a wedge of suspensions of $BP\mathbb{R}$.
- Using the map above, we may identify classes $v_n \in \pi_{(2^n-1)(1+\alpha)} M\mathbb{R}$.

Real spectra derived from $M\mathbb{R}$

- We may now mimic some constructions in non-equivariant chromatic homotopy theory.
- Spectra of interest: Johnson-Wilson $E(n)$, truncated Brown-Peterson $BP\langle n\rangle$

$$\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, \dots, v_n][v_n^{-1}], \qquad \pi_* BP\langle n\rangle = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$$

- From now on, everything is 2-local.
- We may mod out by $v_{n+1}, v_{n+2}, \dots \in \pi_\star BP\mathbb{R}$ to construct Real truncated Brown-Peterson theory, $BP\mathbb{R}\langle n\rangle$.
- We may further invert v_n to construct Real Johnson-Wilson theory, $E\mathbb{R}(n)$.

Facts about $E\mathbb{R}(n)$

- $E\mathbb{R}(n)$ is a (homotopy) associative and commutative ring spectrum.
- $E\mathbb{R}(n)^{C_2} \to E\mathbb{R}(n)^{hC_2}$ is a weak equivalence. (denote these by $ER(n)$)
- There is a fibration

$$\Sigma^{\lambda(n)} ER(n) \to ER(n) \to E(n)$$

  where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. (Kitchloo and Wilson)
- $ER(0) = H\mathbb{Z}_{(2)}$, $ER(1) = KO_{(2)}$

Homotopy of $ER(n)$

Theorem (Hu and Kriz, Kitchloo and Wilson)

$$\pi_* ER(n) = \mathbb{Z}_{(2)}[\mathbf{v}_k(s), \mathbf{v}_n^{\pm 2^{n+1}}, \mathbf{x}]/I, \qquad s \in \mathbb{Z}, 0 \leq k < n$$

where

- $\mathbf{x} \in \pi_{\lambda(n)} ER(n)$ *is 2-torsion* $(\lambda(n) = 2^{2n+1} - 2^{n+1} + 1)$
- $\mathbf{v}_k(s)$ *restricts to* $v_k v_n^{(2^n-1)(2^{k+1}s - 2^k + 1)}$ *in* $E(n)_*$.
- I *detects the relations*

$$\mathbf{v}_0(0) = 2, \qquad \mathbf{x}^{2^{k+1}-1}\mathbf{v}_k(s) = 0$$

- $ER(n)$ *is* $(2^{2n+2} - 2^{n+2})$*-periodic*

Back to the Hurewicz image

- $ER(2)$ has the same homotopy as $TMF\langle 3\rangle$. $TMF\langle 3\rangle$ satisfies a similar fibration. They are conjecturally equivalent.
- So, the Hurewicz image in $ER(1) = KO_{(2)}$ contains η, η^2.
- The Hurewicz image in $ER(2) =_? TMF\langle 3\rangle$ contains $\eta, \nu, \overline{\kappa}$.

$$\vdots$$

- There is a family of spectra $ER(n)$ in which the Hurewicz image seems to grow with n. What is the Hurewicz image in $\pi_* ER(n)$?

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Towards the Hurewicz image in $ER(n)$

- We would like to detect the Hurewicz image on the E_2 page of the Adams spectral sequence for $ER(n)$
- Problem: $ER(n)$ is not connective. It is periodic!
- Solution: Work with a suitable connective model.

Towards the Hurewicz image in $ER(n)$

Plan:

- Find a connective model $er(n)$ of $ER(n)$.
- Compute $H^*(er(n))$ as a module over the Steenrod algebra.
- Run the Adams spectral sequence.

$$E_2^{*,*} = Ext_A^*(H^* er(n), \mathbb{Z}/2) \Longrightarrow \pi_*(er(n))$$

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A connective model of $ER(n)$

Let $BPR\langle n\rangle$ denote $BP\mathbb{R}\langle n\rangle^{C_2}$. Using work of Hu, one can show that

- $\pi_* BPR\langle n\rangle$ contains all of the degree ≥ 0 classes in $\pi_* ER(n)$ that are not multiples of v_n^{-1}
- $BPR\langle n\rangle \to ER(n)$ is an equivalence upon inverting a class in $\pi_* BPR\langle n\rangle$.
- Thus, it deserves to be called a connective model of $ER(n)$.
- **Note 1:** It is *not* a connective cover of $ER(n)$ in the sense that $\pi_k ER(n) \neq \pi_k BPR\langle n\rangle$ in general for $k > 0$. $BPR\langle n\rangle$ removes the "junk" in the connective cover coming from v_n^{-1}.
- **Note 2:** We want $BP\mathbb{R}\langle n\rangle^{C_2}$, not $BP\mathbb{R}\langle n\rangle^{hC_2}$. The homotopy fixed points have negative homotopy.

An unstable splitting and the Hurewicz image

Theorem (Kitchloo and Wilson)

There is an unstable splitting of $\Omega^\infty BP\mathbb{R}\langle n\rangle$ which contains $\Omega^\infty BP\mathbb{R}\langle n-1\rangle$ as a summand.

This splitting can be shown to imply that the Hurewicz image in $BPR\langle n-1\rangle$ is contained in the Hurewicz image in $BPR\langle n\rangle$ (which in turn is in the Hurewicz image in $ER(n)$). Thus, we don't lose any classes as we increase the chromatic height!

Towards $H^*(BPR\langle n \rangle)$

- We can use the Tate diagram.
- Start with the cofiber sequence

$$EC_{2+} \to S^0 \to \widetilde{EC_2} \to \Sigma EC_{2+}$$

- Smash the right three terms with $BP\mathbb{R}\langle n \rangle$ and take fixed points:

$$BP\mathbb{R}\langle n \rangle^{C_2} \to (\widetilde{EC_2} \wedge BP\mathbb{R}\langle n \rangle)^{C_2} \to \Sigma(EC_{2+} \wedge BP\mathbb{R}\langle n \rangle)^{C_2}$$

Towards $H^* BPR\langle n\rangle$

Look at the map of rows induced by $BP\mathbb{R}\langle n\rangle \to BP\mathbb{R}\langle 0\rangle/2$:

$$\begin{array}{ccccc}
BPR\langle n\rangle & \longrightarrow & (\widetilde{EC_2} \wedge BP\mathbb{R}\langle n\rangle)^{C_2} & \longrightarrow & \Sigma(\widetilde{EC_2} \wedge BP\mathbb{R}\langle n\rangle)^{C_2} \\
\downarrow & & \downarrow \varphi_1 & & \downarrow \varphi_2 \\
(BP\mathbb{R}\langle 0\rangle/2)^{C_2} & \longrightarrow & (\widetilde{EC_2} \wedge (BP\mathbb{R}\langle 0\rangle/2))^{C_2} & \longrightarrow & \Sigma(\widetilde{EC_2} \wedge (BP\mathbb{R}\langle 0\rangle/2))^{C_2}
\end{array}$$

Towards $H^*BPR\langle n\rangle$

$$
\begin{array}{ccccc}
BPR\langle n\rangle & \longrightarrow & (\widetilde{EC_2} \wedge BP\mathbb{R}\langle n\rangle)^{C_2} & \longrightarrow & \Sigma(\widetilde{EC_2} \wedge BP\mathbb{R}\langle n\rangle)^{C_2} \\
\downarrow & & \downarrow \varphi_1 & & \downarrow \varphi_2 \\
(BP\mathbb{R}\langle 0\rangle/2)^{C_2} & \longrightarrow & (\widetilde{EC_2} \wedge (BP\mathbb{R}\langle 0\rangle/2))^{C_2} & \longrightarrow & \Sigma(\widetilde{EC_2} \wedge (BP\mathbb{R}\langle 0\rangle/2))^{C_2} \\
\simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\
H\mathbb{Z}/2 & \hookrightarrow & \bigvee_{i=0}^{\infty} \Sigma^i H\mathbb{Z}/2 & \twoheadrightarrow & \bigvee_{i=1}^{\infty} \Sigma^i H\mathbb{Z}/2
\end{array}
$$

- The maps in the bottom row are inclusion of the bottom summand followed by projection onto the higher summands.
- φ_1 and φ_2 are surjections on cohomology
- $(\widetilde{EC_2} \wedge BP\mathbb{R}\langle n\rangle)^{C_2} = \bigvee_{i=0}^{\infty} \Sigma^{i2^{n+1}} H\mathbb{Z}/2$
- To compute $H^*BPR\langle n\rangle$, need to compute φ_1 and φ_2 on cohomology.