Connective covers of Real Johnson-Wilson theories

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Outline

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1. Introduction: detecting homotopy classes

2. Constructing Real spectra

3. Towards the Hurewicz image in $ER(n)$

4. A connective model of $ER(n)$
Introduction: detecting homotopy classes

What classes in $\pi_* S^0$ can we detect in the image of $\pi_* S^0 \to \pi_* E$ (the Hurewicz map) for various cohomology theories $E$?

The image in $\pi_* KO$ contains $\eta$ and $\eta^2$. In fact, all of the torsion in $\pi_* KO$ is in the Hurewicz image.

The image in $\pi_* TMF\langle 3 \rangle$ (topological modular forms with level 3 structure) contains (at least) $\eta$, $\nu$, and $\kappa$. (Mahowald, Rezk)

What is the Hurewicz image in other cohomology theories? Is there a family of cohomology theories which detects more and more of $\pi_* S^0$?
Chromatic homotopy theory

- Let $X$ be a $(p$-local, finite) spectrum. There is a filtration on $X$ such that the $n$th slice is $L_{K(n)}X$, the Bousfield localization of $X$ at Morava $K$-theory, $K(n)$.

- **Theorem:** (Devinatz and Hopkins)

  $$L_{K(n)}S^0 = E_n^{hG(n)}$$

  where $E_n$ denotes Morava $E$-theory, and $G(n)$ denotes the Morava stabilizer group.

- Interesting question: if $G$ is a finite subgroup of $G(n)$, what does $E_n^{hG}$ ‘see’?

- Results in this vein: resolutions of the $K(2)$-local sphere (Goerss, Henn, Mahowald, Rezk), Kervaire invariant one (Hill, Hopkins, Ravenel)
Chromatic homotopy theory

- Let $E(n)$ denote Johnson-Wilson theory, where $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n][v_n^{-1}]$

- One can show $L_{K(n)}E(n) = E_n^{h(\text{Gal}({\mathbb{F}}_{2^n}/\mathbb{F}_2) \times {\mathbb{F}}_{2^n}^\times)}$

- $\pi_* L_{K(n)}E(n)$ is torsion-free, so the Hurewicz image is trivial outside of degree 0.

- But there is a $C_2$-action on $E(n)$ (coming from the $C_2$-action by complex conjugation on $MU$) such that $\pi_* L_{K(n)}E(n)^{hC_2}$ contains torsion.
We would like to compute the Hurewicz image in $\pi_* L_{K(n)} E(n)^{hC_2}$, or better yet $\pi_* E(n)^{hC_2}$.

Morally, $L_{K(n)} E(n)^{hC_2}$ is $\left( E_n^{h(\text{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \times \mathbb{F}_{2^n}^\times)} \right)^{hC_2}$.

Philosophy: control the size of the subgroup of $G(n)$ but work at all chromatic heights.

First, we need to get precise about $E(n)^{hC_2}$. 
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Defining Real cobordism

There is a fiberwise $C_2$-action on the canonical $n$-plane bundle $\gamma_n$ over $BU(n)$. This extends to an action on its Thom space, $\text{Th}(\gamma_n)$.

**Definition**

Let $\alpha$ denote the sign representation of $C_2$. Define

$$M_{\mathbb{R}}(n(1 + \alpha)) = \text{Th}(\gamma_n)$$

Since $\{n(1 + \alpha)\}$ is cofinal among all representations of $C_2$, we may spectrify to obtain a genuine $C_2$-equivariant spectrum, $M_{\mathbb{R}}$.

**Note:** $M_{\mathbb{R}}^*$ is not actually the cobordism ring of Real manifolds.
Facts about $\mathbb{M}R$ (Hu and Kriz)

- $\mathbb{M}R$ is an $E_\infty$-ring spectrum.
- $\mathbb{M}R$ is ‘Real-oriented’. So it supports a formal group law.
- The forgetful map $\mathbb{M}R_* \to MU_*$ is split by a map of rings $MU_* \to \mathbb{M}R_*$ classifying the formal group law, where the image of $x_i$ is in degree $i(1 + \alpha)$.
- There is a Real analog of the Quillen idempotent which produces a $C_2$-equivariant spectrum $BP_R$ such that $\mathbb{M}R(2)$ splits as a wedge of suspensions of $BP_R$.
- Using the map above, we may identify classes $v_n \in \pi(2^n-1)(1+\alpha)\mathbb{M}R$. 

We may now mimic some constructions in non-equivariant chromatic homotopy theory.

- Spectra of interest: Johnson-Wilson $E(n)$, truncated Brown-Peterson $BP\langle n \rangle$

\[ \pi_* E(n) = \mathbb{Z}_p(v_1, \ldots, v_n)[v_n^{-1}], \quad \pi_* BP\langle n \rangle = \mathbb{Z}_p(v_1, \ldots, v_n) \]

- From now on, everything is 2-local.

- We may mod out by $v_{n+1}, v_{n+2}, \ldots \in \pi_* BP\mathbb{R}$ to construct Real truncated Brown-Peterson theory, $BP\mathbb{R}\langle n \rangle$.

- We may further invert $v_n$ to construct Real Johnson-Wilson theory, $E\mathbb{R}(n)$.
Facts about $E_R(n)$

- $E_R(n)$ is a (homotopy) associative and commutative ring spectrum.
- $E_R(n)^{C_2} \to E_R(n)^{hC_2}$ is a weak equivalence. (denote these by $E_R(n)$)
- There is a fibration

$$\Sigma^{\lambda(n)} E_R(n) \to E_R(n) \to E(n)$$

where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. (Kitchloo and Wilson)
- $E_R(0) = H\mathbb{Z}_{(2)}$, $E_R(1) = KO_{(2)}$
Theorem (Hu and Kriz, Kitchloo and Wilson)

\[ \pi_* ER(n) = \mathbb{Z}(2)[v_k(s), v_n^{\pm 2^{n+1}}, x]/I, \quad s \in \mathbb{Z}, 0 \leq k < n \]

where

- \( x \in \pi_{\lambda(n)} ER(n) \) is 2-torsion \( (\lambda(n) = 2^{2n+1} - 2^{n+1} + 1) \)
- \( v_k(s) \) restricts to \( v_k v_n^{(2^n-1)(2^{k+1}s-2^k+1)} \) in \( E(n)_* \).
- \( I \) detects the relations

\[ v_0(0) = 2, \quad x^{2^{k+1}-1} v_k(s) = 0 \]

- \( ER(n) \) is \( (2^{2n+2} - 2^{n+2}) \)-periodic
Back to the Hurewicz image

- $ER(2)$ has the same homotopy as $TMF\langle 3 \rangle$. $TMF\langle 3 \rangle$ satisfies a similar fibration. They are conjecturally equivalent.
- So, the Hurewicz image in $ER(1) = KO_{(2)}$ contains $\eta, \eta^2$.
- The Hurewicz image in $ER(2) = ? TMF\langle 3 \rangle$ contains $\eta, \nu, \eta$.

\[ \vdots \]

- There is a family of spectra $ER(n)$ in which the Hurewicz image seems to grow with $n$. What is the Hurewicz image in $\pi_* ER(n)$?
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Towards the Hurewicz image in $ER(n)$

- We would like to detect the Hurewicz image on the $E_2$ page of the Adams spectral sequence for $ER(n)$
- Problem: $ER(n)$ is not connective. It is periodic!
- Solution: Work with a suitable connective model.
Towards the Hurewicz image in $ER(n)$

Plan:

- Find a connective model $er(n)$ of $ER(n)$.
- Compute $H^*(er(n))$ as a module over the Steenrod algebra.
- Run the Adams spectral sequence.

$$E_2^{*,*} = \text{Ext}_A^*(H^*er(n), \mathbb{Z}/2) \Longrightarrow \pi_*(er(n))$$
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A connective model of $ER(n)$

Let $BPR\langle n \rangle$ denote $BPR\langle n \rangle^C_2$. Using work of Hu, one can show that

- $\pi_* BPR\langle n \rangle$ contains all of the degree $\geq 0$ classes in $\pi_* ER(n)$ that are not multiples of $v_n^{-1}$
- $BPR\langle n \rangle \to ER(n)$ is an equivalence upon inverting a class in $\pi_* BPR\langle n \rangle$.
- Thus, it deserves to be called a connective model of $ER(n)$.

**Note 1:** It is *not* a connective cover of $ER(n)$ in the sense that $\pi_k ER(n) \neq \pi_k BPR\langle n \rangle$ in general for $k > 0$. $BPR\langle n \rangle$ removes the “junk” in the connective cover coming from $v_n^{-1}$.

**Note 2:** We want $BPR\langle n \rangle^C_2$, not $BPR\langle n \rangle^{hC_2}$. The homotopy fixed points have negative homotopy.
Theorem (Kitchloo and Wilson)

There is an unstable splitting of $\Omega^\infty BP \langle n \rangle$ which contains $\Omega^\infty BP \langle n - 1 \rangle$ as a summand.

This splitting can be shown to imply that the Hurewicz image in $BPR \langle n - 1 \rangle$ is contained in the Hurewicz image in $BPR \langle n \rangle$ (which in turn is in the Hurewicz image in $ER(n)$). Thus, we don’t lose any classes as we increase the chromatic height!
Towards $H^*(BPR\langle n\rangle)$

- We can use the Tate diagram.
- Start with the cofiber sequence

$$EC_{2+} \to S^0 \to \widetilde{EC}_2 \to \Sigma EC_{2+}$$

- Smash the right three terms with $BP\mathbb{R}\langle n\rangle$ and take fixed points:

$$BP\mathbb{R}\langle n\rangle^C \to (\widetilde{EC}_2 \wedge BP\mathbb{R}\langle n\rangle)^C \to \Sigma (EC_{2+} \wedge BP\mathbb{R}\langle n\rangle)^C$$
Towards $H^* BPR\langle n \rangle$

Look at the map of rows induced by $BPR\langle n \rangle \rightarrow BPR\langle 0 \rangle/2$:

$$
\begin{align*}
BPR\langle n \rangle & \longrightarrow (\widetilde{EC}_2 \land BPR\langle n \rangle)^{C_2} \longrightarrow \Sigma(\widetilde{EC}_2 \land BPR\langle n \rangle)^{C_2} \\
(BPR\langle 0 \rangle/2)^{C_2} & \longrightarrow (\widetilde{EC}_2 \land (BPR\langle 0 \rangle/2))^{C_2} \longrightarrow \Sigma(\widetilde{EC}_2 \land (BPR\langle 0 \rangle/2))^{C_2}
\end{align*}
$$
Towards $H^* BPR\langle n \rangle$

- The maps in the bottom row are inclusion of the bottom summand followed by projection onto the higher summands.
- $\varphi_1$ and $\varphi_2$ are surjections on cohomology
- $(\widetilde{EC}_2 \wedge BP_R\langle n \rangle)^{C_2} = \bigvee_{i=0}^{\infty} \Sigma^i H\mathbb{Z}/2$
- To compute $H^* BPR\langle n \rangle$, need to compute $\varphi_1$ and $\varphi_2$ on cohomology.