MULTIPLICATIVE STRUCTURE ON REAL JOHNSON-WILSON THEORY

NITU KITCHLOO, VITALY LORMAN, AND W. STEPHEN WILSON

1. Introduction

At the prime 2, Johnson-Wilson theory $E(n)$ [W73] is a complex-oriented cohomology theory which has a $C_2$-equivariant refinement, $\mathbb{E}(n)$ as a genuine $C_2$-equivariant spectrum, where the action of $C_2$ stems from complex conjugation. This was first constructed in [HK01], and Real Johnson-Wilson theory $ER(n)$ is defined to be the $C_2$-fixed points of $\mathbb{E}(n)$. The underlying nonequivariant spectrum of $\mathbb{E}(n)$ is Johnson-Wilson theory $E(n)$, and it is a homotopy associative, commutative, and unital ring spectrum. The goal of this note is to investigate whether the same properties hold of $\mathbb{E}(n)$ and $ER(n)$.

Interest in this problem comes from the fact that $ER(n)$ is quickly becoming a useful and computable cohomology theory. For $n = 1$ and 2, it reproduces familiar cohomology theories, $ER(1) = KO(2)$ and $ER(2) = TMF_0(3)$ (the latter after suitable completion, see [HM15]). The $ER(n)$-cohomology of a large (and growing) collection of spaces has been computed: real projective spaces and their products [KW08a, KW08b, Ban13] (for $n = 2$), complex projective spaces [Lor16, KLW16b], BO and some of its connective covers [KW14, KLW16b], and half of all Eilenberg MacLane spaces [KLW16a, KLW16b]. Furthermore, these computations have applications. In [KW08a, KW08b], the first and third authors used computations in $ER(2)$-cohomology to prove new nonimmersion results for real projective spaces.

The usual nonequivariant techniques that show that $E(n)$ is a ring spectrum do not apply in the Real setting as the obstructions live in groups which are nonzero. However, the existence of a multiplicative structure on Real Johnson-Wilson theory has been suggested in a comment in [HK01] (Comment 5 following the proof of Theorem 2.28) which claims that $\mathbb{E}(n)^*(\mathbb{E}(n) \wedge \mathbb{E}(n))$ may be calculated and from this it may be shown that $\mathbb{E}(n)$ is a (homotopy) associative, commutative, and unital ring spectrum. The results in this note were born in the attempt to verify the above claim. Unfortunately, we were unsuccessful in doing so. However, we do show that $\mathbb{E}(n)$ represents an $\mathbb{M}U$-algebra which is homotopy unital, associative, and commutative up to phantom maps. By a phantom map, we mean a map $f : X \rightarrow Y$ which has trivial restriction to any finite CW complex mapping into $X$. In addition, we show that the $\mathbb{E}(n)$-cohomology of an equivariant topological space is a commutative ring.

Date: September 12, 2016.
The first author is supported in part by the NSF through grant DMS 1307875.
Theorem 1.1. $\mathbb{E}(n)$ is a homotopy commutative, homotopy associative, unital real $\mathbb{MU}$-algebra up to phantom maps. In other words, there exist unit and multiplication maps:

$$1 : \mathbb{MU} \longrightarrow \mathbb{E}(n), \quad \hat{\mu} : \mathbb{E}(n) \land \mathbb{E}(n) \longrightarrow \mathbb{E}(n),$$

such that all the obstructions to $\hat{\mu}$ being a homotopy associative and homotopy commutative $\mathbb{MU}$-algebra structure are phantom maps. Differently said, all the corresponding structure diagrams commute up to phantom maps. Furthermore, the forgetful map:

$$\rho : \mathbb{E}(n)^0(\mathbb{E}(n) \land \mathbb{E}(n)) \longrightarrow E(n)^0(E(n) \land E(n)),$$

maps $\hat{\mu}$ to the canonical product $\mu$ on the non-equivariant Johnson-Wilson spectrum $E(n)$.

Theorem 1.1 tells us that the Real Johnson-Wilson theory is valued in commutative rings when applied to finite CW complexes. Our second result extends this to the category of all spaces.

Theorem 1.2. With any choice of multiplication $\hat{\mu}$ as above, the spectrum $\mathbb{E}(n)$ represents a cohomology theory on the category of $C_2$-spaces valued in (bigraded) commutative rings. There are natural transformations of ring-valued cohomology theories $\mathbb{MU}^*(-) \longrightarrow \mathbb{E}(n)^*(-)$ and $\mathbb{E}(n)^*(*) \longrightarrow E(n)^*(*)$.

The results of this document justify the assumption of commutativity in the computations of the $ER(n)$-cohomology of topological spaces made by the authors in previous work.

The authors would like to thank Neil Strickland for a helpful discussion related to this document.

2. Background

In this section, we recall a few background definitions and theorems from [KW07b] and [KLUW16b] that we use in subsequent sections.

A Real (or genuine $C_2$-equivariant) spectrum $\mathbb{E}$ is a family of $C_2$-spaces $\mathbb{E}_{a+ba}$, indexed over elements $a + b\alpha \in RO(C_2)$ where $\alpha$ denotes the sign representation, together with a compatible system of equivariant homeomorphisms

$$\mathbb{E}_{a-r+(b-s)\alpha} \xrightarrow{\sim} \Omega^{r+s\alpha} \mathbb{E}_{a+b\alpha}$$

where the right hand side denotes the space of pointed maps (endowed with the conjugation action) from the one point compactification of the representation $r + s\alpha$. The reader may refer to [HK01] for more details on Real spectra. We will denote by $ER$, the homotopy fixed point spectrum of the $C_2$-action on $\mathbb{E}$. A canonical example of a Real spectrum is $\mathbb{MU}$, whose underlying nonequivariant spectrum is complex cobordism, $MU$, studied first by Landweber [Lan68], Fujii [Fuj76], Araki and Murayama [AM78], and more recently by Hu-Kriz [HK01]. The action of $C_2$ is induced by the complex conjugation action on the pre-spectrum representing $\mathbb{MU}$ in the usual way.
The $p = 2$ Johnson-Wilson theory $E(n)$ lifts to a Real spectrum, $E(n)$, defined as an $\mathbb{M}\mathbb{U}$-module by coning off certain equivariant lifts of the Araki generators $v_i$ for $i > n$, and then inverting the lift of $v_n$. We shall call these equivariant lifts by the same names, $v_i$. The Real Johnson-Wilson theories, $ER(n)$, are defined as the fixed points of $E(n)$.

Working with cohomological grading, let $Y$ be a $C_2$-space and let $E^{(1+\alpha)}_*(Y)$ denote the sub-ring of diagonal elements in the equivariant $E$-cohomology of $Y$ i.e.

$$E^{(1+\alpha)}_*(Y) := \pi_0\text{Maps}_{C_2}(Y, E^{(1+\alpha)}_*)$$

Consider the ring homomorphism given by forgetting the $C_2$-action:

$$\rho : E^{(1+\alpha)}_*(Y) \to E_*(Y).$$

Notice that the image of $\rho$ belongs to the graded sub-group of elements in even degree.

**Definition 2.1.** A $C_2$-space $Y$ is said to have the weak projective property with respect to a real spectrum $E$ if the map $\rho$:

$$\rho : E^{(1+\alpha)}_*(Y) \to E_*(Y),$$

is an isomorphism of graded abelian groups.

**Definition 2.2.** A $C_2$-space $X$ is said to be projective if

1. $H_*(X; \mathbb{Z})$ is of finite type.
2. $X$ is homeomorphic to $\bigvee I (\mathbb{C}P^\infty)^\wedge k_I$ for some weakly increasing sequence of integers $k_I$, with the $C_2$ action given by complex conjugation.

By a $C_2$-equivariant $H$-space, we shall mean an $H$-space whose multiplication map is $C_2$-equivariant.

**Definition 2.3.** A $C_2$-equivariant $H$-space $Y$ is said to have the projective property if there exists a projective space $X$, along with a pointed $C_2$-equivariant map $f : X \to Y$, such that $H_*(Y; \mathbb{Z}/2)$ is generated as an algebra by the image of $f$.

**Theorem 2.4.** Let $Y$ be a $C_2$-equivariant $H$-space with the projective property. Let $E$ denote any complete Real $\mathbb{M}\mathbb{U}$-module spectrum with underlying spectrum $E$, satisfying the property that the forgetful map: $\rho^* : E^{(1+\alpha)}_* \to E_*$, is an isomorphism. Then the space $Y$ has the weak-projective property with respect to $E$. In other words, the following map is an isomorphism of $\mathbb{M}\mathbb{U}^{(1+\alpha)}$-modules:

$$\rho : E^{(1+\alpha)}_*(Y) \to E_*(Y).$$

**Remark 2.5.** The smash product of a finite collection of spaces with the projective property is an example of a space that has the weak projective property, but not the projective property.
Spaces with the projective property are not rare because many spaces have homology generated by the images of elements coming from complex projective spaces. The example of interest in this appendix will be $E(n)_0$ and its products.

**Lemma 2.6.** $E(n)_0^{\times j} \times \mathbb{MU}_i^{\times s}_{(2^n-1)(1+\alpha)}$ has the weak projective property with respect to $E(n)$ for all $i, j, s \geq 1$.

**Proof.** By Theorem 1-4 of [KW13], $E(n)_0^{\times j} \times \mathbb{MU}_i^{\times s}_{(2^n-1)(1+\alpha)}$ is a restricted product of a family of spaces with the projective property (i.e. it is the colimit of finite products of spaces with projective property). A product of spaces with projective property is evidently projective, and since $\rho: E(n)^{(1+\alpha)}(-) \to E(n)^{2s}(-)$ is an isomorphism at each stage, it follows that it is an isomorphism in the limit. \qed

### 3. A stable Multiplicative structure

We begin with the observation that the spaces $E(n)_0$ are $(2^n-1)(1+\alpha)$-periodic. Hence we may express the spectrum $E(n)_0$ and its products as colimits:

$$E(n)_0 = \lim colim_m \Sigma^{-(2^n-1)(1+\alpha)} E(n)_0, \quad E(n)^{\wedge k}_0 = \lim colim_m \Sigma^{-(2^n-1)(1+\alpha)} E(n)^{\wedge k}_0,$$

where the maps in the inverse system are given by successive $E(n)^{\wedge k}_0$ multiplication:

$$\Sigma^{-(2^n-1)(1+\alpha)} E(n)^{\wedge k}_0 \to \Sigma^{-(2^n-1)(1+\alpha)} E(n)^{\wedge k}_0.$$

We now apply $E(n)_0$ cohomology. Milnor’s $\lim^1$-sequence gives us

$$0 \to \lim \lim^1 E(n)_0^{-1}(E(n)^{\wedge k}) \to E(n)_0^0(E(n)^{\wedge k}) \to \lim \lim^0 E(n)_0^0(E(n)^{\wedge k}) \to 0,$$

where we have used the $(2^n-1)(1+\alpha)$-periodicity of $E(n)_0$ to identify $E(n)_0^0(\Sigma^{-(2^n-1)(1+\alpha)} E(n)_0)$ with $E(n)_0^0(\Sigma^{-(2^n-1)(1+\alpha)} E(n)_0)$.

We may now invoke the weak projective property of the spaces $E(n)^{\wedge k}_0$, and identify the last term with $\lim \lim^0 E(n)_0^0(E(n)^{\wedge k})$. This inverse limit can be seen to be $E(n)_0^0(E(n)^{\wedge k})$ via an analogous (non-equivariant) Milnor sequence in which the $\lim^1$ term vanishes since the $E(n)_0$-cohomology of $E(n)_0$ is evenly graded. Thus we obtain the short exact sequence:

$$0 \to \lim \lim^1 E(n)_0^{-1}(E(n)^{\wedge k}) \to E(n)_0^0(E(n)^{\wedge k}) \to E(n)_0^0(E(n)^{\wedge k}) \to 0,$$

with the last map being $\rho$.

By writing $E(n)^{\wedge k}$ as a colimit of $v_n^{k}$-multiplication maps as above and $\mathbb{MU}^s$ as a colimit of suspension spectra

$$\mathbb{MU}^s = \lim m \Sigma^{m-s(2^n-1)(1+\alpha)} \mathbb{MU}^s_{m(2^n-1)(1+\alpha)}$$

the above proof readily extends to show the existence of a short exact sequence

$$0 \to \lim \lim^1 E(n)_0^{-1}(E(n)^{\wedge r}) \mathbb{MU}^s_{m(2^n-1)(1+\alpha)}) \to E(n)_0^0(E(n)^{\wedge r}) \mathbb{MU}^s \to \cdots$$
\[ \cdots \rightarrow E(n)^0(E(n)^k \wedge MU^s) \rightarrow 0. \]

where we have used Lemma 2.6 above to identify the right hand term.

We are now ready to construct the \( MU \)-algebra structure on \( E(n) \) that will be shown to be homotopy commutative and homotopy associative up to phantom maps.

**Definition 3.1.** Define \( \hat{\mu} \) to be any element in \( E(n)^0(E(n)^{\wedge 2}) \) that lifts the canonical ring structure of \( E(n)^0(E(n)^{\wedge 2}) \) along \( \rho \). Define the unit map \( 1 : MU \rightarrow E(n) \) to be the canonical map expressing \( E(n) \) as a quotient of \( MU \).

As a formal consequence of the short exact sequence constructed above, we obtain Theorem 1.1 from the introduction:

**Theorem 1.1.** The class \( \hat{\mu} \) defines a homotopy commutative, homotopy associative, unital real \( MU \)-algebra structure on \( E(n) \) up to phantom maps.

**Proof.** By construction, \( \hat{\mu} \) maps to the canonical ring structure on \( E(n) \) under the forgetful map, \( \rho \). It follows that any obstruction to the homotopy associativity, commutativity, or unitality of \( \hat{\mu} \), viewed as a class in \( E(n)^0(E(n)^{\wedge r} \wedge E(n)^{\wedge s}) \), maps to zero under \( \rho \). We claim that the only elements of the kernel of \( \rho \) are phantoms.

To see this, notice that the inclusion of the \( \lim^1 \)-term in \( E(n)^0(E(n)^{\wedge r} \wedge MU^s) \) restricts trivially to all the terms \( E(n)^0(E(n)^{\wedge r} \wedge MU^s) \). Since any map from a finite CW complex into \( E(n)^{\wedge r} \wedge MU^s \) must factor through a finite stage of the colimit, it follows that the image of the \( \lim^1 \) term in \( E(n)^0(E(n)^{\wedge r} \wedge MU^s) \) consists entirely of phantoms. \( \square \)

**Remark 3.2.** One may attempt to compute the group of phantom maps \( \lim^1 E(n)^{-1}(E(n)^{\wedge k}) \) explicitly by identifying it with the vector space \( E(n)^{-1}(E(n)^{\wedge k}) \otimes \mathbb{Z}/2 \) (suitably extended by a \( \mathbb{Z}/2 \)-algebra). This is an open problem, but it appears to the authors that this vector space is trivial for \( n = 1 \), but may fail to be so for \( n > 1 \). Hence we at present have no general way of ensuring that the ring structure we have constructed is rigid up to homotopy for the spectra \( E(n) \), \( n > 1 \).

4. **Multiplicative structure on \( MU(2)[v_n^{-1}] \)**

We would like to show that the multiplication \( \hat{\mu} \) constructed above naturally induces a commutative algebra structure on the \( E(n) \)-cohomology of any space. An essential ingredient in our construction will be the multiplication on \( MU(2)[v_n^{-1}] \). We pause to describe it in this section. The ingredient we need is the following proposition, which appears as Proposition 9.15 in [HHR09] and is proved in [HH13].
Proposition 4.1. \([HH13]\) Let \(R\) be a \(G\)-equivariant commutative ring with \(D \in \pi_G^*(R)\). If \(D\) has the property that for every \(H \subset G\), \(N_H^G \nu^*_H D\) divides a power of \(D\), then the spectrum \(D^{-1}R\) has a unique commutative algebra structure such that the map \(R \rightarrow D^{-1}R\) is a map of commutative rings.

We begin by constructing a \(C_2\)-equivariant associative and commutative ring (in the highly structured sense) that lifts \(E(n)\). We begin with \(\mathbb{MU}\) which has this structure (see \([HHR09]\) or \([HK01]\)). We localize at \(p = 2\). The spectrum \(\mathbb{MU}(2)\) is a \(C_2\)-equivariant commutative ring, as shown in \([HH13]\). By \([HK01]\), the forgetful map \(\rho : \pi_*(1+\alpha)(\mathbb{MU}(2)) \rightarrow \pi_2^*(MU(2))\) is an isomorphism. The classes \(v_i\) (Araki, Hazewinkel, or others) in \(\pi_2^*(MU(2))\) may now be lifted via \(\rho^{-1}\) to equivariant classes, \(\rho^{-1}(v_i)\). While in the rest of the manuscript, we abuse notation by denoting \(\rho^{-1}(v_i)\) by \(v_i\), in the following lemma, we will distinguish between the nonequivariant \(v_i\) and the equivariant \(\rho^{-1}(v_i)\).

Our next step is to invert \(\rho^{-1}(v_n)\).

Lemma 4.2. The spectrum \((\rho^{-1}(v_n))^{-1}\mathbb{MU}(2)\) is a \(C_2\)-equivariant commutative ring.

Proof. We apply Proposition 4.1 above (quoted from \([HHR09]\)). The map \(i^*_H\) is exactly \(\rho\), and so \(\rho(\rho^{-1}(v_n)) = v_n \in \pi_2(2^{n-1})(\mathbb{MU}(2))\). We need to show that \(N_{\{e\}}^{C_2}(v_n)\) divides a power of \(\rho^{-1}(v_n)\). In fact, we claim that \(N_{\{e\}}^{C_2}(v_n) = -[\rho^{-1}(v_n)]^2\). To see this, we apply the isomorphism \(\rho\) to both sides. The double coset formula (see e.g. Proposition 10.9(v) in \([Sch]\) or \([May96]\)) reduces in our case to

\[
\rho \circ N_{\{e\}}^{C_2}(v_n) = v_n \cdot c(v_n) = -v_n^2
\]

which completes the proof. \(\square\)

Remark 4.3. Let us again denote the spectrum \((\rho^{-1}(v_n))^{-1}\mathbb{MU}(2)\) by \(\mathbb{MU}[v_n^{-1}]\). This spectrum serves as a commutative proxy for \(E(n)\). Indeed, essentially all results that hold for \(E(n)\) extend verbatim to this spectrum. In particular, results in \([KW07a, KW07b, KW08a, KW08b, Ban13, KW13, Lor16, KLW16a, KLW16b]\) hold with \(E(n)\) replaced by \(\mathbb{MU}[v_n^{-1}]\).

5. Unstable properties of the Multiplicative structure

We now address the question of the (unstable) multiplicative structure on \(E(n)\). We begin with the following lemma:

Lemma 5.1. There is a unique (homotopy) commutative, associative and unital equivariant H-ring structure on the infinite loop space of \(E(n)\) that lifts the H-ring structure on \(E(n)_0\):

\[
\hat{\mu}_0 : E(n)_0 \times E(n)_0 \rightarrow E(n)_0
\]
Proof. Recall that Lemma 2.6 shows that $E(n)^{X_j}$ has the weak projective property. As $E(n)$ is a (homotopy) associative and commutative ring spectrum, we may define the map $\mu_0 : E(n)_0 \times E(n)_0 \to E(n)_0$ as the preimage of the multiplication on $E(n)_0$ along the isomorphism

$$E(n)^0(E(n)_0 \times E(n)_0) \cong E(n)^0(E(n)_0 \times E(n)_0)$$

The unity, commutativity, associativity of $\mu_0$ are similarly verified by applying the isomorphism $\rho$, as the desired relations all hold in $E(n)$-cohomology. \hfill \Box

Remark 5.2. The unstable multiplication $\mu_0$ we constructed in the previous lemma and the stable multiplication $\mu$ we constructed in Definition 3.1 are compatible in the sense that $\mu$ restricts to $\mu_0$ on the zero space of $E(n)$. To see this, apply the isomorphism $\rho$ and note that this claim is true nonequivariantly by construction.

We now prove our second main result, Theorem 1.2.

Proof. (of Theorem 1.2) Let $X$ be a space and consider $f \in E(n)^V(X)$ and $g \in E(n)^W(Y)$. We define the product $fg \in E(n)^{V+W}(X)$ as follows. First, note that since $E(n)$ is an $\mathbb{MU}[v_n^{-1}]$-module, we may multiply $f$ and $g$ by classes in the coefficients $\mathbb{MU}[v_n^{-1}]$. Let $k$ and $l$ be the minimal integers such that

$$v_n^k f \in E(n)^V(X), \quad v_n^l g \in E(n)^W(X)$$

with $V', W' \leq 0$ (by this we mean that when we express each representation as a combination of irreducibles, each coefficient should be nonpositive). These classes are represented by maps

$$X \xrightarrow{\varepsilon_n f} E(n)_V = \Omega^-V'E(n)_0, \quad X \xrightarrow{\varepsilon_n g} E(n)_W = \Omega^-W'E(n)_0$$

We adjoin the loops over to form classes

$$\Sigma^{-V'}X \xrightarrow{v_n^k f} E(n)_0, \quad \Sigma^{-W'}X \xrightarrow{v_n^l g} E(n)_0$$

Note that since these are positive suspensions, the sources of these maps are spaces. We may thus smash them together, precompose with the diagonal on $X$ and postcompose with the multiplication on the zero space constructed in Lemma 5.1.

$$\Sigma^{-V'-W'}X \xrightarrow{\Delta} \Sigma^{-V'}X \times \Sigma^{-W'}X \xrightarrow{\varepsilon_n f \times \varepsilon_n g} E(n)_0 \times E(n)_0 \xrightarrow{\mu_0} E(n)_0$$

This produces a class in $E(n)^{V+W'}(X)$. Finally, we multiply by $v_n^{-k-l}$ to define the product $\mu n \cdot g \in E(n)^{V+W}(X)$.

The unit of this multiplication comes from the unit on $\mathbb{MU}[v_n^{-1}]$. The unity, associativity, and commutativity of this multiplication follow from the corresponding properties of $E(n)_0$.  

7
We have shown that for any space $X$, $E(n)^*(X)$ is a graded associative and commutative ring. It remains to show that this is compatible with the multiplication on $\mathbb{MU}[v_n^{-1}]$-cohomology. If we carry out the above construction to define multiplication on $\mathbb{MU}[v_n^{-1}]^*(X)$, it is evident that this agrees with the multiplication coming from the ring spectrum structure on $\mathbb{MU}[v_n^{-1}]$. To see that this multiplication agrees with the one defined on $E(n)$-cohomology, it suffices to check this fact on zero spaces. To see that this diagram

$$
\begin{array}{ccc}
\mathbb{MU}_0 \times \mathbb{MU}_0 & \xrightarrow{\mu_{\mathbb{MU}}} & \mathbb{MU}_0 \\
\downarrow & & \downarrow \\
E(n)_0 \times E(n)_0 & \xrightarrow{\mu_{E(n)}} & E(n)_0
\end{array}
$$

commutes, we may use the fact that $\mathbb{MU}_0$ and $\mathbb{MU}_0 \times \mathbb{MU}_0$ are spaces with weak projective properties to map the diagram isomorphically along $\rho$ where its commutativity is apparent. □

REFERENCES


Department of Mathematics, Johns Hopkins University, Baltimore, USA

*E-mail address:* nitu@math.jhu.edu

*E-mail address:* vlorman@math.jhu.edu

*E-mail address:* wsw@math.jhu.edu