Math 108 Midterm 1 Practice

Print Name: ___________________________  Section: ______________

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ___________________________  Date: ______________

• This is a 50 minute closed book exam. No notes, books, or calculators are allowed.

• Present your solution to each problem in a clear and orderly fashion. Show all your work. An answer without justification will not receive full credit.

• Do not use any techniques we have not covered in class yet.

• This exam contains 8 pages (including this cover page) and 5 questions. The last page is intended for use as scrap paper.

The table on the right is for grading purposes. Please do not write in it.

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Total:</td>
<td>140</td>
<td></td>
</tr>
</tbody>
</table>
1. Determine whether each one of the following is TRUE or FALSE. If the statement is false, explain why or give a counterexample.

(a) (5 points) The function $f(x) = \sin(x)$ is one-to-one.

**Solution:** Since $f(0) = f(\pi) = 0$, $f$ is not one-to-one.

(b) (5 points) If a function $f(x)$ is continuous at $a$, then it is differentiable at $a$.

**Solution:** The function $f = |x|$ is continuous at 0 but not differentiable at 0.

(c) (5 points) If $f$ and $g$ are differentiable functions, then $(fg)' = f'g'$.

**Solution:** Let $f(x) = x$ and $g(x) = x$. Then $fg = x^2$ and we have

$$(fg)'(x) = 2x \neq 1 \cdot 1 = f'(x) \cdot g'(x)$$

(d) (5 points) If $f$ is a continuous function and $\lim_{x \to 4} f(x) = 5$, then $f(4) = 5$.

**Solution:** Since $f$ is continuous, $f(a) = \lim_{x \to a} f(x)$.

(e) (5 points) If $f$ and $g$ are continuous functions, then $f + g$ is a continuous function.

**Solution:** True
2. Evaluate the following limits justifying each step.

(a) (15 points) \[ \lim_{x \to 2} \frac{x^2 - 4}{x^2 - 6x + 8} \]

**Solution:** We have

\[
\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 6x + 8} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{(x - 2)(x - 4)} \\
= \lim_{x \to 2} \frac{x + 2}{x - 4} \\
= \frac{2 + 2}{2 - 4} = \frac{4}{-2} = -2
\]

where (2) follows from the fact that \( \frac{(x - 2)(x + 2)}{(x - 2)(x - 4)} = \frac{x + 2}{x - 4} \) when \( x \neq 2 \) and (3) follows from the fact that \( \frac{x + 2}{x - 4} \) is a continuous function on its domain.

Points earned: _____ out of 15 points
(b) (15 points) \( \lim_{x \to \infty} \sqrt{x^2 + 3x + 1} - x \)

Solution: We have

\[
\lim_{x \to \infty} \sqrt{x^2 + 3x + 1} - x = \lim_{x \to \infty} \left( \sqrt{x^2 + 3x + 1} - x \right) \cdot \frac{\sqrt{x^2 + 3x + 1} + x}{\sqrt{x^2 + 3x + 1} + x} \tag{5}
\]

\[
= \lim_{x \to \infty} \frac{(x^2 + 3x + 1) - x^2}{\sqrt{x^2 + 3x + 1} + x} \tag{6}
\]

\[
= \lim_{x \to \infty} \frac{3x + 1}{\sqrt{x^2 + 3x + 1} + x} \cdot \frac{1}{x} \tag{7}
\]

\[
= \lim_{x \to \infty} \frac{3 + \frac{1}{x}}{\sqrt{1 + \frac{3}{x} + \frac{1}{x^2} + 1}} \tag{8}
\]

\[
= \lim_{x \to \infty} \frac{3 + \frac{1}{x}}{\sqrt{1 + \frac{3}{x} + \frac{1}{x^2} + 1}} \cdot \lim_{x \to \infty} \left( \frac{3}{x} \right) \tag{9}
\]

\[
= \lim_{x \to \infty} \frac{3 + \frac{1}{x}}{\sqrt{1 + \frac{3}{x} + \frac{1}{x^2} + 1}} + \lim_{x \to \infty} 1 \tag{10}
\]

\[
= \frac{3}{\sqrt{1 + 1}} = \frac{3}{2} \tag{11}
\]

Here, (9) follows from the quotient law for limits, (10) follows from the sum law and the law involving the square root, and (11) follows from the fact that \( \lim_{x \to \infty} \frac{1}{x^a} = 0 \) for \( a > 0 \).
3. (a) (15 points) Let \( f, g \) be continuous functions defined on \([1, 3]\) such that \( f(1) < g(1) \) and \( f(3) > g(3) \). Show that there exists a number \( c \) in the interval \((0, 3)\) such that \( f(c) = g(c) \).

**Solution:** Let \( h(x) = f(x) - g(x) \). We have that \( h \) is a continuous function on the interval \([1, 3]\), being the difference of continuous functions. Since \( f(1) < g(1) \), we have

\[
h(1) < 0.
\]

Analogously, since \( f(3) > g(3) \), we have

\[
h(3) > 0.
\]

By the Intermediate Value Theorem, there exists a number \( c \) in the interval \((1, 3)\) so that \( h(c) = 0 \). For this \( c \), we have \( f(c) = g(c) \).

(b) (15 points) Find all asymptotes (horizontal and vertical) of \( f(x) = \sqrt{2x^6 + x^4 + 3} \).

**Solution:** The graph of \( f \) will have a vertical asymptote, at the points where the denominator vanishes as long as the numerator is non-zero at that point. Factoring the denominator, we get

\[
x^3 + x^2 - 2x = x(x + 2)(x - 1)
\]

therefore the denominator vanishes at \( x = 0, 1, -2 \). The numerator \( \sqrt{2x^6 + x^4 + 3} \) is always positive and therefore, in particular, does not vanish at \( x = 0, 1, -2 \). The vertical asymptotes are therefore \( x = 0, x = 1 \) and \( x = -2 \).

To find horizontal asymptotes, we need to compute \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \). We have

\[
\lim_{x \to \infty} \frac{\sqrt{2x^6 + x^4 + 3}}{x^3 + x^2 - 2x} = \lim_{x \to \infty} \frac{\sqrt{2x^6 + x^4 + 3}}{x^3 + x^2 - 2x} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}
\]

\[
= \lim_{x \to \infty} \frac{\frac{1}{x^3}(2x^6 + x^4 + 3)}{\frac{1}{x^3}(x^3 + x^2 - 2x)}.
\]

\[
= \lim_{x \to \infty} \frac{2 + \frac{1}{x^2} + \frac{3}{x^4}}{1 + \frac{1}{x} - \frac{2}{x^2}} = \frac{\sqrt{2}}{1}
\]
and

\[ \lim_{x \to -\infty} \frac{\sqrt{2x^6 + x^4 + 3}}{x^3 + x^2 - 2x} = \lim_{x \to -\infty} \frac{\sqrt{2x^6 + x^4 + 3} \cdot \frac{1}{x^3}}{x^3 + x^2 - 2x} \cdot \frac{1}{x^3} = \lim_{x \to -\infty} -\frac{\sqrt{\frac{1}{x^6}(2x^6 + x^4 + 3)}}{\frac{1}{x^3}(x^3 + x^2 - 2x)} = \lim_{x \to -\infty} -\frac{\sqrt{2 + \frac{1}{x^6} + \frac{3}{x^2}}}{1 + \frac{1}{x} - \frac{2}{x^3}} = -\sqrt{2} \]

where we have used the fact that when \( x < 0 \), we have \( x\sqrt{A} = -\sqrt{x^2 \cdot A} \) for any \( A \). The horizontal asymptotes are therefore \( y = \sqrt{2} \) and \( y = -\sqrt{2} \).
4. (a) (15 points) Find the derivative of \( f(x) = \sqrt{x^2 + 3} \) using the definition of derivative.

**Solution:** We have

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(x + h)^2 + 3} - \sqrt{x^2 + 3}}{h} \cdot \frac{\sqrt{(x + h)^2 + 3} + \sqrt{x^2 + 3}}{\sqrt{(x + h)^2 + 3} + \sqrt{x^2 + 3}} \\
&= \lim_{h \to 0} \frac{(x + h)^2 + 3 - (x^2 + 3)}{h(\sqrt{(x + h)^2 + 3} + \sqrt{x^2 + 3})} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h(\sqrt{(x + h)^2 + 3} + \sqrt{x^2 + 3})} \\
&= \lim_{h \to 0} \frac{2x + h}{\sqrt{(x + h)^2 + 3} + \sqrt{x^2 + 3}} \\
&= \frac{2x}{\sqrt{x^2 + 3}} \\
&= x \sqrt{x^2 + 3}
\end{align*}
\]

(b) (10 points) Find the equation of the tangent line to the graph of \( f \) at the point \((1, 2)\).

**Solution:** The derivative of \( f \) at 1 is

\[
f'(1) = \frac{1}{\sqrt{1^2 + 3}} = \frac{1}{2}.
\]

The equation of the tangent line is therefore

\[
(y - 2) = \frac{1}{2}(x - 1)
\]

or equivalently

\[
y = \frac{1}{2}x + \frac{3}{2}.
\]
5. (a) (15 points) Find the derivative of \( f(x) = \frac{x^2 + 2x + 1}{x+2} \)

**Solution:** Using the quotient rule, we have

\[
f'(x) = \frac{(x + 2)(2x + 2) - (x^2 + 2x + 1)(1)}{(x + 2)^2}
\]

\[
= \frac{x^2 + 4x + 3}{(x + 2)^2}
\]

(b) (15 points) Find the values of \( A \) and \( B \) such that the function

\[
\begin{cases} 
  x^2 + 1 & x < 0 \\
  A \sin x + B \cos x & x \geq 0
\end{cases}
\]

is differentiable.

**Solution:** For \( f \) to be differentiable, \( f \) has to be continuous. We have

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 + 1 = 1
\]

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} A \sin x + B \cos x = B
\]

so \( f \) can only be continuous at 0 if \( B = 1 \). We set \( B = 1 \). We then have \( \lim_{x \to 0^-} f(x) = 1 \) and since \( f(0) = 1 \), the function \( f \) is continuous at 0. We still need to find the value of \( A \) so that \( f \) is differentiable at 0. We have

\[
\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(h)^2 + 1 - (0)^2 + 1}{h} = \frac{d}{dx} (x^2 + 1)|_{x=0} = (2x)|_{x=0} = 0
\]

and

\[
\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(A \sin h + \cos h) - (A \sin 0 + \cos 0)}{h} = \frac{d}{dx} (A \sin x + \cos x)|_{x=0} = (A \cos x - \sin x)|_{x=0} = A
\]

Therefore the limit \( \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} \) exists only if \( A = 0 \). Therefore \( B = 1 \) and \( A = 0 \).
This page is intended for use as scrap paper.