MIDTERM

1.) Let \( g(x) = \frac{x^2 + 3x + 2}{x^2 + 5x + 6} \).

a.) All \( x \) except \( x = -2, -3 \).

b.) Everywhere except \( x = -2 \) and \( x = -3 \) since \( g(x) \) is a rational function.

c.) \( \lim_{x \to -2} \frac{x^2 + 3x + 2}{x^2 + 5x + 6} = \lim_{x \to -2} \frac{(x + 2)(x + 1)}{(x + 2)(x + 3)} \)

\[ = \lim_{x \to -2} \frac{x + 1}{x + 3} \]

\[ = \frac{-2 + 1}{-2 + 3} \]

\[ = -1. \]

d.) The degrees of the top and bottom are the same, so the limit is the coefficients of the leading terms. Thus, the limit is \( \frac{1}{1} = 1 \).

e.) \( g'(x) = \frac{(x^2 + 5x + 6)(2x + 3) - (x^2 + 3x + 2)(2x + 5)}{(x^2 + 5x + 6)} \).

2.) \( f(x) = 500 + 10x - 5x^2 \).

a.) \( f(x + h) = 500 + 10(x + h) - 5(x + h)^2 \)

\[ = 500 + 10x + 10h - 5(x^2 + 2xh + h^2) \]

\[ = 500 + 10x + 10h - 5x^2 - 10xh - 5h^2. \]

So

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

\[ = \lim_{h \to 0} \frac{500 + 10x + 10h - 5x^2 - 10xh - 5h^2 - (500 + 10x - 5x^2)}{h} \]

\[ = \lim_{h \to 0} \frac{10h - 10xh - 5h^2}{h} \]

\[ = \lim_{h \to 0} 10 - 10x - 5h \]

\[ = 10 - 10x. \]

b.) \( f'(x) = 10 - 10x \). The derivative is zero when \( x = 1 \). If \( x < 1 \) then \( f'(x) > 0 \); otherwise, \( f'(x) < 0 \). So the function is increasing if \( x < 1 \) and decreasing if \( x > 1 \).

c.) The only local extremum is zero. By examining the first derivatives, we
see that it is a local maximum.

d.) \( f''(x) = -10 \). So the function is always concave down.

e.) See graph.

f.) I have no idea.

3.) Find the following limits or explain why they are undefined:

a.) \((-1)^5 + (-1)^2 - 1 = -1 + 1 - 1 = -1\).

b.) \( \lim_{t \to 2} \frac{t + 1}{t - 2} = \infty \) since both the top and bottom are positive.

c.) \( \lim_{x \to -\infty} \frac{x^4 + x + 1}{x^2 + x + 1} = \lim_{x \to -\infty} \frac{x^4(1 + \frac{1}{x} + \frac{1}{x^4})}{x^2(1 + \frac{1}{x} + \frac{1}{x^2})} = \lim_{x \to -\infty} x^2(1 + \frac{1}{x} + \frac{1}{x^4}) = (\infty)^2(\frac{1}{0} + 0) = (\infty)^2(-\frac{1}{2}) = -\infty. \)

d.) We check the limit from each side. First, if \( x > 0 \) then \(|x| = x\)

\[ \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1. \]

If \( x < 0 \) then \(|x| = -x\). So

\[ \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1. \]

Since the two sides aren’t the same, the limit does not exist.

e.) If the limit is not defined, the function is not continuous.

4.) See diagram for figure. We have the equation

\[ x^2 + y^2 = 5^2. \]

If we take the derivative of both sides,

\[ 2x \, dx + 2y \, dy = 0. \]
Dividing by $dt$ gives

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.$$ 

Now, we have that $x = 3$ and $\frac{dy}{dt} = -1$. We can use the first equation to find that $y = \sqrt{5x^2 - 3^2} = 4$. So we wish to find $\frac{dx}{dt}$. Plugging in and solving:

$$2(3) \frac{dx}{dt} + 2(4)(-1) = 0$$
$$6 \frac{dx}{dt} = 8$$
$$\frac{dx}{dt} = \frac{8}{6} = \frac{4}{3}.$$

5.) Miscellany:

a.) Note that $f(x + h) = |x + h|$. At $x = 0$, this means that $f(0) = 0$, $f(x + h) = |h|$. So

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{|h| - 0}{h}$$
$$= \lim_{h \to 0} \frac{|h|}{h}.$$

As was in the case of problem 3 part d, this limit is not defined. So the function is not differentiable.

b.) $f(-x) = (-x)^3 + 171(-x) = -x^3 - 171x = -f(x)$. So $f$ is odd.

c.) $\cos(240^\circ) = -\frac{1}{2}$.

d.) $h(0) = -5$, $h(2) = 7$. By IVT, since $h(0) < 0 < h(2)$, there is a point $x$ on $[0, 2]$ such that $h(x) = 0$. Additionally, since $h$ is continuous and differentiable, by mean value theorem, there is another point $c$ on $[0, 2]$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{7 - (-5)}{2 - 0} = \frac{12}{2} = 6.$$

6.) Find the following derivatives:

a.) $f'(x) = x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x) = x \cos x + \sin x$.

b.) By chain rule, let $f(r) = \sqrt{r}$, $g(r) = \sec x + 12$. So

$$f'(r) = \frac{1}{2\sqrt{r}}$$
$$f'(g(r)) = \frac{1}{2\sqrt{\sec x + 12}}$$
\( g'(r) = \sec x \tan x. \)

So

\[ h'(r) = f'(g(r)) \cdot g'(r) = \left( \frac{1}{2\sqrt{\sec x + 12}} \right) \sec x \tan x. \]

c.) \( g'(x) = e^{3x^2+24x+6}(6x + 24) \) (by chain rule or by the exponential rule)

d.) \( y = (\tan x)^x \) means

\( \ln y = \ln(\tan x)^x \)

\( \ln y = x \ln(\tan x). \)

Taking implicit derivative on both sides,

\[ \frac{1}{y} \cdot dy = x \cdot \frac{1}{\tan x} \cdot \sec^2 x \cdot dx + \ln(\tan x) \cdot dx. \]

Dividing both sides by \( dx \),

\[ \frac{1}{y} \cdot \frac{dy}{dx} = x \left( \frac{1}{\tan x} \cdot \sec^2 x \right) + \ln(\tan x). \]

Solving for \( \frac{dy}{dx} \),

\[ \frac{dy}{dx} = [x \left( \frac{1}{\tan x} \cdot \sec^2 x \right) + \ln(\tan x)]y. \]

Finally, we plug in \( (\tan x)^x \) for \( y \):

\[ \frac{dy}{dx} = [x \left( \frac{1}{\tan x} \cdot \sec^2 x \right) + \ln(\tan x)](\tan x)^x. \]