Prime Gaps and Adeles

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History

2000 years ago, Euclid posed the following conjecture:

**Twin Prime Conjecture:** There are infinitely many primes \( p \) such that \( p + 2 \) is also prime.

If \( p \) and \( p + 2 \) are prime, the pair are called twin primes.

This was generalized in 1849 by Alphonse de Polignac:

**Polignac Conjecture:** For any even \( k \), there are infinitely many primes \( p \) such that \( p + k \) is also prime.

If \( p \) and \( p + k \) are prime, we will call them \( k \)-familial primes.

(Of course, this definition is only useful for \( k \) even.)
These aren't the only conjectures involving pairs of primes:

If $p$ and $2p + 1$ are both prime, the pair are called **Germain Primes**.

It is conjectured that there are infinitely many.

In fact, for any positive integer $n$, it is conjectured that there infinitely many primes $p$ such that $2^n p + 1$ is also prime.

A similar statement can be made for pairs of primes of the form $(p, 2^n p - 1)$.

If $p$ and $2^n p \pm 1$ are prime, we will call them **1-familial primes** (modulo powers of two, they only differ by one).

In other words, 1-familial primes are things that look like Germain Primes.
Hardy-Littlewood Conjectures

All of these conjectures were concatenated and made more exact by Hardy and Littlewood.

Let $\pi_k(x)$ be the number of pairs of $k$-familial primes less than $x$.

**Hardy-Littlewood Conjecture:** Let $k \geq 2$ be an even integer. As $x$ becomes large,

$$\pi_k(x) \sim C_k \frac{x}{\log^2 x},$$

where $C_k$ is an explicitly computable constant, known as the Hardy-Littlewood constant.

For 1-familial primes, there is a similar story:

Define:

- $\pi_{1,n}^+(x) = \{ \#p \leq x : p \text{ and } 2^np + 1 \text{ are both prime} \}$,
- $\pi_{1,n}^-(x) = \{ \#p \leq x : p \text{ and } 2^np - 1 \text{ are both prime} \}$.

**Hardy-Littlewood Conjecture for 1-familial primes:** Let $n \in \mathbb{N}$. As $x$ becomes large,

$$\pi_{1,n}^+(x) \sim \pi_{1,n}^-(x) \sim C_2 \frac{x}{\log^2 x},$$

where $C_2$ is as defined above.
Relation Between Hardy-Littlewood Conjectures?

Proving these conjectures is hard.

However, perhaps we can prove them (conditionally) to be equivalent.

**Theorem 1** (Weiss-W. 2009): Assume that
\[ \pi_{2^j}(x) \sim C_{2^j} \frac{x}{\log^2 x} \]
for \( j = 1, 2, ..., n - 1 \), and assume that
\[ \pi_{1,m}^+(x) \sim \pi_{1,m}^+(x) \sim C_2 \frac{x}{\log^2 x} \]
as conjectured. Moreover, assume that these \( 2^j \)-familial primes are "well-distributed across congruence classes" (for \( 0 \leq j \leq n - 1 \)). Then there are infinitely many \( 2^n \) familial primes.

In other words, if, for \( j = 0, 1, ..., n - 1 \), the \( 2^j \)-familial primes are of conjectured density and are well-distributed then there are infinitely many \( 2^n \) familial primes.

For example, if the conjectured density of 1, 2, and 4-familial primes is correct and these primes are "well-distributed" then there are infinitely many 8-familial primes.
Notes

For notation’s sake, write that for \( k \geq 2 \), \( a \in T_k \) if \( a \) and the \( a + k \) are \( k \)-familial.

Moreover, for \( k = 1 \), write that \( a \in T_1 \) if \( a = 2^n p \pm 1 \) for some prime \( p \).

By well-distributed, we mean that for some parameter that \( m \) is approximately \( \sqrt{Z} \), we have that for \( \sigma, \delta > 0 \),
\[
\sum_{a = Z^{1/2-\sigma}\mu}^{Z^{1/2-\sigma}\mu} \frac{\sin(\frac{a\pi}{m})}{a} \leq \frac{1}{Z^{\mu}}
\]
for some \( \mu > 0 \).

Why you should believe this: If the above sum were changed to be over all primes in this range (instead of only those in \( T_k \)), the inequality would simply a corollary of GRH.

**Theorem 1a**: If the above inequality holds for all \( k \), the Hardy-Littlewood conjectures are equivalent for all \( k = 2^n \) with \( n \geq 1 \).
New Methods: P-adic Numbers

Instead of using traditional sieve methods, we use the language of the $p$-adics.

For $x \in \mathbb{Q}$, define the $p$-adic absolute value by

$$|x|_p = \begin{cases} p^{-\text{ord}_p x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\mathbb{Q}_p = (\text{analytic}) \text{ completion of } \mathbb{Q} \text{ with respect to the } p\text{-adic absolute value}.$

Also, add in $\mathbb{Q}_\infty = \mathbb{R}$.

When all of these fields are stapled together, they are known as the adeles; as such, the methods here are known as adelic methods.

Unlike $\mathbb{Q}$, these fields allow for nice analysis.
New Methods (cont’d)

We adapt methods that have been developed for Diophantine equations.

In our case, for \( k \geq 2 \), the goal is to find prime solutions to the equation

\[
f(x) = x_1 - x_2 = k.
\]

We define a weighting function \( \phi \) that will give primes a larger weight than composites.

The function is defined place by place over all of the \( p \)-adics (including \( \mathbb{Q}_\infty \)) and then stapled together, i.e. we define \( \phi_v \) for all \( v \) and then let

\[
\phi(x_1, x_2) = \prod_v \phi_v(x_1, x_2).
\]
The Real Place

Let $Z$ be a large integer, and let

$$Y = Z^{1-\delta},$$
$$V = Z^{\frac{1}{4}-\epsilon}.$$

In the real place,

$$\phi_\infty(x_1, x_2) = e^{-\frac{\pi x_1^2}{2}}.$$

The real place is the Gaussian function (i.e. bell curve), which decays quickly.

As such truncates the area over which we have to look (if $x_1$ and $x_2$ are large, $\phi_\infty$ is very small).
The \( p \)-adic Places (For Small \( p \))

In the \( p \)-adic places where \( p \leq Y \),

\[
\phi_p(x_i) = \begin{cases} 
1 & \text{if } x_i \in \mathbb{Z}_p - p\mathbb{Z}_p \text{ (i.e. } p \nmid x_1), \\
\frac{1}{V} & \text{if } x_1 \in p\mathbb{Z}_p - p^2\mathbb{Z}_p \text{ (i.e. } p\| x_1) \\
0 & \text{otherwise (i.e. } p^2 \mid x_i \text{ or } x_i \text{ not an integer).}
\end{cases}
\]

and

\[
\phi_p(x_1, x_2) = \phi_p(x_1)\phi_p(x_2).
\]

These check to see whether \( p \mid x_1 \) or \( x_2 \). Each time one of these primes divides \( x_1 \) or \( x_2 \), the function attaches an weight of \( \frac{1}{V} \) (if \( x_1 \) and \( x_2 \) aren’t square-free, the weight is zero).

As such, if \( x_1 \) and \( x_2 \) are primes, they will have weight approximately \( \frac{1}{V^2} \). If \( x_1 \) or \( x_2 \) are composite, the weight will be \( \frac{1}{V^3} \) or less.
The $p$-adic Places (For Large $p$)

In the $p$-adic places where $p > Y$,

$$\phi_p(x_i) = \begin{cases} 
1 & \text{if } x_i \in \mathbb{Z}_p \text{ (i.e. } x_1 \text{ is an integer)}, \\
0 & \text{otherwise}.
\end{cases}$$

If one of these primes divides $x_1$ or $x_2$ then $x_1$ and $x_2$ are large, which means that the real place has already eliminated them anyway.

As such, we don’t care about these primes.
Let

\[ N(k) = \sum_{x_1, x_2 \in \mathbb{Q}, x_1 - x_2 = k} \phi(x_1, x_2). \]

This gives an approximation for the number of pairs of primes that differ by \( k \).

So \( N(k) \) is a weighted sum which will approximate the number of \( k \)-familial primes up to \( \sqrt{Z} \).

Notably, \( \phi(x_1, x_2) \) is a nice (i.e. Schwartz) function, meaning that the tools of analysis (especially Fourier analysis) can be used.

Using all manner of analytic tricks, we use this setup to compare \( N(k) \) for the various \( k \), eventually resulting in Theorem 1.