On the Connection Between Igusa Local Zeta Functions and Generalized Exponential Sums

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Abstract

This paper gives a formula for Weil’s Exponential Sums which depends entirely upon the singularities mod $\pi$, similar to the purpose of the Stationary Phase Formula for Igusa Local Zeta Functions. The paper then shows why the newly discovered formula for Generalized Exponential Sums does not correspond more closely with the Stationary Phase Formula for Igusa Local Zeta Functions by showing that the Stationary Phase Formula’s requirement that one consider points where the function evaluates to zero modulo $\pi$ is a result of the transformations and not necessarily the Exponential Sum itself. In proving this, the paper gives a simplification of the transformation necessary to find an Igusa Local Zeta Function from its associated Generalized Exponential Sum. 1

1 Introduction

In the early 1970’s, Jun-Ichi Igusa, in an attempt to better understand the behavior of polynomials, began the study of what became known as Igusa Local Zeta Functions. Denoted $Z(s)$, they are defined as follows:

$$Z(s) = \int_{O_K} |f(\bar{x})|_K^s d\bar{x}$$

where $K$ is a $\pi$-adic completion of a number field $k$, $| \cdot |_K$ is the $\pi$-adic valuation, and $f(\bar{x}) \in O_K[x_1, x_2, \ldots, x_n]$.

It was discovered in 1994 by Igusa in [5] that there exists an organizing method, known as the Stationary Phase Formula (or SPF). The Stationary Phase Formula expresses the Igusa Local Zeta Function in terms of the singular points (the points where all of the partial derivatives evaluates to zero) modulo $\pi$, as well as the points where the function itself evaluates to zero mod $\pi$:

**SPF for Igusa Local Zeta Functions**: For some $\pi$-adic field $K$ and $f(\bar{x}) \in O_K[x_1, x_2, \ldots, x_n]$, let

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\[ N = \{ a \in \mathcal{O}_K / \pi \mathcal{O}_K \mid f(a) \equiv 0 \pmod{\pi} \} \]

\[ S = \{ a \in N \mid \partial f / \partial x_i(a) \equiv 0 \pmod{\pi} \text{ for } 1 \leq i \leq n \} \]

\[ p = [\mathcal{O}_K : \pi \mathcal{O}_K] \]

\[ t = p^{-s} \]

Then

\[ Z(t) = (p^n - |N|)p^{-n} + (|N| - |S|)p^{-n}t(1 - p^{-1}) + \sum_{a \in S} \int_{\mathcal{O}_K} |f(\vec{x})|_K d\vec{x} \]

Interestingly, Igusa Local Zeta Functions bear a close link to another important number theoretic function, the exponential sum. While exponential sums have existed in number theory for many years, this particular exponential sum, known as the Generalized Exponential Sum, Weil’s Exponential Sum, or Generalized Gaussian Sum was developed largely by Weil in the 1940’s as a method of attacking certain types of zeta functions (as discussed in [8]). It is defined as

\[ F^*(i^*) = \int_{\mathcal{O}_K^n} \Psi(i^* f(\vec{x})) d\vec{x} \]

where \( \Psi(\vec{x}) \) is a character in \( \text{Hom}(K, \mathbb{C}^\times) \) such that

\[ \Psi|_{\mathcal{O}_K} = 1, \quad \Psi|_{\pi^{-e} \mathcal{O}_K} \neq 1 \quad \forall \ e > 0 \]

and \( i^* = \pi^{-e} u \) for \( u \in \mathcal{O}_K^\times \).

The connection between these two functions has been known for a long time; in fact, there exists an explicit method of transforming an exponential sum into a local zeta function. This method, described in [6], requires two steps.

Let us begin with a Generalized Exponential Sum for some \( f(\vec{x}) \in \mathcal{O}_K[x_1, x_2, \ldots, x_n] \).

The first step is to use an inverse Fourier Transform to find what is known as the Local Singular Series, generally denoted \( F(i) \). The meaning of such a series will not be expounded upon here, except as a go-between for our transformation; for our purposes, it will be defined in terms of the exponential sum and then summarily ignored:

\[ F(i) = \int_K F^*(i^*) \psi(-ii^*) d i^* \]

The second step takes our singular series to a local zeta function using a Mellon Transform

\[ Z(s) = \int_{\mathcal{O}_K} F(i) |i|_K^{-s} d i \]

Because the Igusa Local Zeta Function and the Generalized Exponential Sum are so closely linked, it would logically follow that one could find an organizing principle for the exponential sum similar to SPF for local zeta functions. In fact, we show in Section 2 that, assuming \( e > 1 \), this is almost the case:
Theorem 2 (SPF for Generalized Exponential Sums): Let

\[ S = \{ a \in O_k / \pi O_k | \frac{\partial f}{\partial x_i} (a) \equiv 0 \pmod{\pi} \text{ for } 1 \leq i \leq n \} \]

Then

\[ F^*(i^*) = \sum_{a \in S} \int_{a + \pi O_k} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n \]

Note that this organizing principle requires only the points where the partial derivatives vanish and not the points where the function itself is congruent to zero. This is because the Generalized Exponential Sum does not make such a distinction. As an example, if one took \( K = \mathbb{Q}_3, i^* = 3^{-2} \), and \( f(x) = x^2 + x \), the value of \( F^*(i^*) \) comes entirely from the points where \( x \equiv 1 \pmod{3} \), even though the function itself does not vanish at these points.

It stands to reason that the organizing principles of the Igusa Local Zeta Function must appear either as a result of the transforms taken upon the Generalized Exponential Sum or as an organizing principle for the exponential sum itself. Clearly, the fact that SPF for local zeta functions examines points where \( f(\bar{x}) \equiv 0 \pmod{\pi} \) is not the result of the exponential sum itself. One would assume, then, that the examination of such points is a result of the transforms. In section 4, we find that this is indeed the case by proving the following theorem:

Theorem 4: Define

\[ N_\epsilon = \{ a \in O_k / \pi^\epsilon O_k | f(a) \equiv 0 \pmod{\pi^\epsilon - 1}, f(a) \not\equiv 0 \pmod{\pi^\epsilon} \} \]
\[ N'_\epsilon = \{ a \in O_k / \pi^\epsilon O_k | f(a) \equiv 0 \pmod{\pi^\epsilon} \} \]
\[ M(s) = \int_{O_k} |x|^s K \]

Then

\[ Z(s) = M(s) + (M(s) - p^s) \sum_{c=1}^{\infty} (p^{-\epsilon - cs - cn}) |p^{\epsilon - 1}(p - 1)|N'_\epsilon| - p^{-\epsilon}|N_\epsilon| \]

Note that if \( \epsilon = 1 \) then

\[ N_1 = \{ f(a) \not\equiv 0 \pmod{\pi} \} \]
\[ N'_1 = \{ f(a) \equiv 0 \pmod{\pi} \} \]

which is exactly the distinction we were missing.

The requirement of the intermediary step in the transformation from \( F^*(i^*) \) to \( Z(s) \) seems rather unwieldy; as it turns out, it is also unnecessary. In proving theorem 4, we discover that the formulation of \( Z(s) \) in terms of \( F^*(i^*) \) can be simplified; in Section 3 of this paper, we show that the step finding \( F(i) \) can be eliminated, and \( Z(s) \) can be written explicitly in terms of \( F^*(i^*) \):
Theorem 3.2: Let $M(s)$ be as above. Then

$$Z(s) = M(s) + (M(s) - p^s)[\int_K |z^s| K^{s+1} F^d(z^s)dz^s - M(s+1)]$$

2 Singular Points and Exponential Sums

In this section, we show that the examination of points whose partial derivatives all evaluate to zero mod $\pi$ is an organizing principle for Generalized Exponential Sums.

Theorem 2: Let $F^*(\alpha) = \int_{O^\pi K} \Psi(\alpha f(x_1, x_2, ..., x_n))dx_1dx_2...dx_n$

where $\alpha = \pi^{-\nu}u$. Further, let $\bar{a}$ be a vector such that $\bar{a} \in (O_K/\pi O_K)^n$. We define $S$ such that $\bar{a} \in S$ if $\partial f/\partial x_i(\bar{a}) \equiv 0 \ (mod \ \pi)$ for all $i$. If $e > 1$ then

$$F^*(\alpha) = \sum_{\bar{a} \in S} \int_{\bar{a} + \pi O_K} \Psi(\alpha f(x_1, x_2, ..., x_n))dx_1dx_2...dx_n.$$

Proof: We can break $F^*(\alpha)$ into a sum of integrals of the vectors which are not singular points mod $\pi$ plus a sum of integrals over the vectors which are singular points mod $\pi$ as follows:

$$F^*(\alpha) = \sum_{\bar{a} \notin S} \int_{\bar{a} + \pi O_K} \Psi(\alpha f(x_1, x_2, ..., x_n))dx_1dx_2...dx_n + \sum_{\bar{a} \in S} \int_{\bar{a} + \pi O_K} \Psi(\alpha f(x_1, x_2, ..., x_n))dx_1dx_2...dx_n.$$

Now, we must show that the first integral is zero. By a change of variables

$$\int_{\bar{a} + \pi O_K} \Psi(\alpha f(x_1, x_2, ..., x_n))dx_1dx_2...dx_n = \int_{O_K} \Psi((\alpha - \nu) f(x_1, x_2, ..., x_n))dx_1dx_2...dx_n$$

which, by Taylor’s Theorem, is equal to

$$\int_{O_K} \Psi((\alpha - \nu) f(x_1, x_2, ..., x_n))dx_1dx_2...dx_n$$

since $\Psi$ is an additive character. By assumption, one of the partial derivatives must be congruent to a unit modulo $\pi$. So let us define $\partial f/\partial x_i(\bar{a}) = c_i \forall i$ and let $c_j$ be a partial which is a unit mod $\pi$ for some $j$. We change variables as follows:

$$x_i = x_i \forall i \neq j$$
This change of variables is known to be measure invariant (as was shown in [6]). Using this,

\[
\int_{\mathcal{O}_K} \Psi((\pi^{-e}u) \sum_{i=1}^n \pi x_i \frac{\partial f}{\partial x_i}(\vec{a}) + \pi^2(...)) dx_1 dx_2 ... dx_n \\
= \int_{\mathcal{O}_K} \Psi((\pi^{-e+1}u)x_j') dx_j' dx_2' ... dx_n'
\]

Clearly,

\[
\int_{\mathcal{O}_K} \Psi(\pi^{-e+1}ux_j) dx_j = 0
\]

and thus

\[
\sum_{\vec{a} \notin S} \int_{\mathcal{O}_K} \Psi(\pi^{e}f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n = 0. \quad \Box
\]

3 A Simplified Transform from \( F^*(i^*) \) to \( Z(s) \)

In this section, we discover a new formula which can be used to transform a Generalized Exponential Sum into an Igusa Local Zeta function. First, though, we need a lemma about the relations between roots of unity:

**Lemma 3.1**: Let \( a \in \mathcal{O}_K \), \( u \in \mathcal{O}_K^\times \). Let \( a' \) be the first nonzero term in the \( \pi \)-adic expansion of \( au \), and let \( k, e \in \mathbb{N} \). If \( k \leq e-1 \), then

\[
\sum_{a \in \pi^k \mathcal{O}_k - \pi^{k+1} \mathcal{O}_k} \Psi(-\pi^{-e} au) = \begin{cases} 
-1 & \text{if } e-1 = k \\
0 & \text{if } e-1 > k
\end{cases}
\]

**Proof**: First, let \( \zeta_k \) denote a \( k \)-th root of unity. So \( \Psi(-\pi^{-e} au) = \zeta_k^{a'} \).

Note that \( \pi \nmid a' \). So

\[
\sum_{a \in \pi^k \mathcal{O}_k - \pi^{k+1} \mathcal{O}_k} \Psi(-\pi^{-e} au) = \sum_{pl|a'} \zeta_k^{a'}
\]

Moreover, note that \( \sum_{pl|a'} \zeta_k^{a'} = -\sum_{a'=1}^{p^{e-k-1}} \zeta_k^{a'} \), since

\[
\sum_{pl|a'} \zeta_k^{a'} = \sum_{a'=1}^{p^{e-k-1}} \zeta_k^{a'} + \sum_{a'=1}^{p^{e-k-1}} \zeta_k^{a'}
\]

\[
= \sum_{pl|a'} \zeta_k^{a'} + \sum_{|a'|} \zeta_k^{a'}
\]

\[
= \sum_{a'=1}^{p^{e-k-1}} \zeta_k^{a'}
\]

So

\[
\sum_{a \in \pi^k \mathcal{O}_k - \pi^{k+1} \mathcal{O}_k} \Psi(-\pi^{-e} au)
\]


\[
\sum_{k=1}^{p^{e-1}} c_k^{p^{e-1}} \\
= \begin{cases} 
-1 & \text{if } e-1 = k \\
0 & \text{if } e-1 > k
\end{cases}
\]

Now we can write an explicit formula for \( Z(s) \) in terms of \( F^*(i^*) \):

**Theorem 3.2:** Let \( M(s) = \int_{O_K} |x|^s d\bar{x} \). Then

\[
Z(s) = M(s) + (M(s) - p^s)[\int_K |i^*|^{e-1} F^*(i^*) di^* M(s + 1)]
\]

**Proof:** Recall that the zeta function can be described in terms of the Local Singular Series:

\[
Z(s) = \int_{O_K} F(i)|i|^s d\bar{i}
\]

where \( F(i) \) can in turn be expressed in terms of the Generalized Exponential Sum:

\[
F(i) = \int_K F^*(i^*) \psi(-i^* d\bar{i}^*) = \int_K \int_{O_K} \psi(f(\bar{x})i^*) \psi(-i^*) d\bar{d}^* di^*
\]

Putting these together, we see that

\[
Z(s) = \int_{O_K} \int_K \int_{O_K} \psi(f(\bar{x})i^*) \psi(-i^*) |i|^s d\bar{d}^* di^* d\bar{i}
\]

Note that \( O_K \) is of finite measure and \( K \) is sigma-finite. So we can change the order of integration. Moreover, we can break up the integral over \( K \) such that:

\[
\int_K = \int_{O_K} + \sum_{e=1}^{\infty} \int_{\pi^{-e} O_K - \pi^{-e+1} O_K}
\]

So \( Z(s) \) can be rewritten as

\[
\int_{O_K} \int_{O_K} \int_{O_K} \psi(f(\bar{x})i^*) \psi(-i^*) |i|^s d\bar{d}^* d\bar{i}^* d\bar{i}
\]

where the last step follows from the fact that \( \psi(f(\bar{x})i^*) \psi(-i^*) = 1 \forall i^* \in O_K \). Clearly, the first summand is \( M(s) \). So we need to show that the second summand is equal to

\[
(M(s) - p^s)[\int_K |i^*|^{e-1} F^*(i^*) di^* M(s + 1)]
\]

To show that this is the case, we break up the integral over \( O_K \):
\[ \int_{O_K} \sum_{i} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \int_{\pi O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \] 
\[ = \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \sum_{k=0}^{e-1} \int_{\pi^k O_K - \pi^k O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]
\[ + \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \int_{\pi O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]

For the latter triple integral, \( \frac{1}{\mid i^* \mid K} > \mid i^* \mid K \). So \( \psi(-i^*) = 1 \), which means that

\[ \int_{O_K} \sum_{i} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \int_{\pi O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]
\[ = \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]
\[ = \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]
\[ = \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]
\[ = \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} M(s) p^{-e-es} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]

where the antepenultimate step is a change of variables. For the former triple integral, we note that the innermost integral can be broken up as follows:

\[ \sum_{k=0}^{e-1} \int_{\pi^k O_K - \pi^k O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \]
\[ = \sum_{k=0}^{e-1} \sum_{a \in (O_K / \pi^{k-1} O_K)^*} \int_{\pi^a O_K - \pi^a O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \]
\[ = \sum_{k=0}^{e-1} \sum_{a \in (O_K / \pi^{k-1} O_K)^*} \int_{\pi^a O_K - \pi^a O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \]

Changing the measure of the inside integral and pulling out the constants gives

\[ \sum_{k=0}^{e-1} \sum_{a \in (O_K / \pi^{k-1} O_K)^*} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \]
\[ = \sum_{k=0}^{e-1} \sum_{a \in (O_K / \pi^{k-1} O_K)^*} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \]
\[ = \psi(f(\vec{x})^*) \sum_{k=0}^{e-1} \sum_{a \in (O_K / \pi^{k-1} O_K)^*} \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \]
\[ = \psi(f(\vec{x})^*) \sum_{k=0}^{e-1} \sum_{a \in (O_K / \pi^{k-1} O_K)^*} \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \]

where the penultimate step is by Lemma 3.1, since \( i^* \in \pi^{-1} O_K - \pi^{-1} O_K \).

Putting these together,

\[ \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \int_{\pi O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]
\[ = \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \int_{\pi O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]
\[ = \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-1} O_K - \pi^{-1} O_K} \int_{\pi O_K} \psi(f(\vec{x})^* \psi(-i^* \vec{x}) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} \]

Now, note that:

\[ \int_{O_K} \int_{O_K} \psi(f(\vec{x})^*) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} = \int_{O_K} \int_{O_K} \psi(f(\vec{x})^*) \mid \vec{x} \mid K^* d\vec{d} \, d\vec{\bar{d}} = M(s + 1) \]
\[ \int_K - \int_{O_K} = \sum_{e=1}^{\infty} \int_{\pi^{-e} O_K - \pi^{-e+1} O_K} \]

So

\[
(M(s) - p^s) \int_{O_K} \sum_{e=1}^{\infty} \int_{\pi^{-e} O_K - \pi^{-e+1} O_K} \psi(f(\tilde{x})) \frac{|i^*|^{1+s}}{K} \, d\tilde{x} \, di^* \\
= (M(s) - p^s) [\int_{O_K} \int_K \psi(f(\tilde{x})) |i^*|^{1+s} \, d\tilde{x} \, dx - M(s + 1)] \\
= (M(s) - p^s) [\int_K \int_{O_K} \psi(f(\tilde{x})) |i^*|^{1+s} \, d\tilde{x} \, di^* - M(s + 1)] \\
= (M(s) - p^s) [\int_K F^*(i^*) |i^*|^{1+s} \, di^* - M(s + 1)] \]

4 Transforms and Zeroes modulo \( \pi \)

In this section, we find that the need to examine the points where \( f(\tilde{x}) \) evaluates to zero modulo \( \pi \) is the result of the transforms undertaken to go from and Exponential Sum to a Zeta Function.

First, let us define sets \( N_e \) and \( N'_e \) such that \( a \in N_e \) implies that \( a \in O_K/\pi O_K, f(a) \equiv 0 \) (mod \( \pi^{e-1} \)) but \( \neq 0 \) (mod \( \pi^e \)), and \( a \in N'_e \) implies \( a \in O_K/\pi O_K, f(a) \equiv 0 \) (mod \( \pi^e \)).

**Theorem 4:** Let \( M(s), N_e, N'_e \) be as above. Then

\[
Z(s) = M(s) + (M(s) - p^s) \sum_{e=1}^{\infty} (p^{-e-\varepsilon - en}) |p^{-e-1}(p - 1)| N'_e \\
- p^{-e} |N_e| \]

**Proof:** From the previous section, we know that

\[
Z(s) = M(s) + (M(s) - p^s) \sum_{e=1}^{\infty} (p^{-e-\varepsilon - en}) \int_{\pi^{-\varepsilon} O_K - \pi^{-\varepsilon+1} O_K} \int_{O_K} \psi(i^* f(\tilde{x})) d\tilde{x} di^* \\
\]

Splitting up the integral over \( O_K \),

\[
\int_{\pi^{-\varepsilon} O_K - \pi^{-\varepsilon+1} O_K} \sum_{a \in O_K/\pi^e O_K} \int_{\pi^e O_K} \psi(i^* f(\tilde{x})) d\tilde{x} di^* \\
= \int_{\pi^{-\varepsilon} O_K - \pi^{-\varepsilon+1} O_K} \sum_{a \in O_K/\pi^e O_K} \int_{\pi^e O_K} \psi(i^* f(a) + \pi^e \cdot \footnote{\( \cdots \)}) d\tilde{x} di^* \\
= \int_{\pi^{-\varepsilon} O_K - \pi^{-\varepsilon+1} O_K} \sum_{a \in O_K/\pi^e O_K} p^{-en} \int_{O_K} \psi(i^* f(a)) d\tilde{x} di^* \\
= \sum_{a \in O_K/\pi^e O_K} p^{-en} \int_{O_K} \int_{\pi^{-\varepsilon} O_K - \pi^{-\varepsilon+1} O_K} \psi(i^* f(a))) d\tilde{x} di^* \\
\]

If \( \pi^k || f(a) \) where \( k < e - 1 \) then by Lemma 3.1,

\[
\int_{\pi^{-\varepsilon} O_K - \pi^{-\varepsilon+1} O_K} \psi(i^* f(a)) di^* = 0 \\
\]

Likewise, if \( k = e - 1 \) then by Lemma 3.1,

\[
\int_{O_K} \int_{\pi^{-\varepsilon} O_K - \pi^{-\varepsilon+1} O_K} \psi(i^* f(a)) di^* dx = -p^{-e} \int_{O_K} dx = -p^{-e} \\
\]

If \( \pi^e || f(a) \) then \( \psi(0) = 1 \), meaning that the integral is simply the measure

8
of the area of integration:

\[
\int_{O_K}^{\pi - \epsilon O_K} \psi(0) di^* dx = p^e \left( \frac{p-1}{p} \right) \int_{O_K}^{\pi - \epsilon + 1} dx = p^e \left( \frac{p-1}{p} \right)
\]

Combine these and we find that

\[
\sum_{a \in O_K / \pi O_K} p^{-en} \int_{O_K}^{\pi - \epsilon O_K} \psi(i^*(f(a))) di^* dx
\]

\[
= p^{-en} \left( \sum_{a \in O_K / \pi O_K, \pi^{-1} | f(a) - p^{-e}} + \sum_{a \in O_K / \pi O_K, \pi^{-1} | f(a)} p^{e \left( \frac{p-1}{p} \right)} \right)
\]

\[
= -p^{-en-\epsilon} |N_a| + p^{-en+e^{-1}(p-1)} |N'_a|
\]

from which our theorem follows.

\[\square\]

References


