ON THE CONNECTION BETWEEN IGUSA LOCAL ZETA FUNCTIONS AND GENERALIZED EXPONENTIAL SUMS

THOMAS WRIGHT

Abstract. This paper considers a formula for Weil’s Generalized Exponential Sums which depends entirely upon the singularities modulo \( \pi \), similar to the purpose of the Stationary Phase Formula (SPF) for Igusa Local Zeta Functions (ILZF). Unlike the formula discussed in this paper, the SPF for ILZF also requires one to examine the points where the function evaluates to zero modulo \( \pi \); this paper shows where the disparity between the two formulae arises. In proving this, the paper gives a simplification of the transformation necessary to find an ILZF from its associated Generalized Exponential Sum.

1. Introduction

In the early 1970’s, Jun-Ichi Igusa, in an attempt to better understand the behavior of polynomials, began the study of what became known as Igusa Local Zeta Functions (or ILZF). Denoted \( Z(s) \), they are defined as follows:

\[
Z(s) = \int_{O_K} |f(x)|_K^s dx,
\]

where \( K \) is a non-archimedean local field of characteristic zero with valuation \( \cdot |_K \), \( O_K \) is the ring of integers in \( K \), \( x = (x_1, \ldots, x_n)\), \( dx \) is the Haar measure on \( K \) normalized such that the measure of \( O_K \) is 1, and \( f(x) \in O_K[x_1, x_2, \ldots, x_n] \). We denote \( \pi O_K \) to be the maximal ideal in \( O_K \) and note that \( O_K/\pi O_K \cong \mathbb{F}_q \).

Igusa Local Zeta Functions functions are generating functions which count the number of solutions to \( f(x) \equiv 0 \) modulo \( \pi \), \( \pi^2 \), \( \pi^3 \), etc. Naturally, such a quantity bears relation to other important mathematical ideas; for example, it can be used to write Poincare sums explicitly [2], and it was instrumental in proving the convergence of Siegel-Eisenstein series [7].

It was discovered in 1994 by Igusa in [6] that there exists an organizing principle for the Igusa Local Zeta Function; one can evaluate an ILZF by examining the singular points (the points where all of the partial derivatives of \( f \) evaluate to zero) modulo \( \pi \), as well as the points where \( f \) itself evaluates to zero modulo \( \pi \). This organizing principle is codified in what is known as the Stationary Phase Formula (or SPF) for Igusa Local Zeta Functions:

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SPF for Igusa Local Zeta Functions. For some $p$-adic field $K$ and $f(x) \in O_K[x_1, x_2, ..., x_n]$, let

$$N = \{a \in O_K/\pi O_K | f(a) \equiv 0 \mod \pi\},$$

$$S = \{a \in N | \frac{\partial f}{\partial x_i}(a) \equiv 0 \mod \pi \text{ for } 1 \leq i \leq n\},$$

$$q = [O_K : \pi O_K]$$

$$t = q^{-s}.$$

Then

$$Z(t) = (q^n - |N|)q^{-n} + (|N| - |S|)q^{-n}t(1 - q^{-1}) + \sum_{a \in S} \int_{a + \pi O_K} |f(x)|_K^s dx.$$  

Interestingly, Igusa Local Zeta Functions bear a close link to another important number theoretic function, the exponential sum. While exponential sums have existed in number theory for many years, this particular exponential sum, known as the Generalized Exponential Sum, Weil’s Exponential Sum, or Generalized Gaussian Sum was developed largely by Weil in the 1940’s in considering certain types of zeta functions (as discussed in [10]). It is defined as

$$F^* = \int_{O_K^*} \Psi(f(x))dx,$$

where $\Psi(x)$ is a character in the dual $K^*$ of $K$ such that

$$\Psi|_{O_K} = 1, \Psi|_{\pi^{-e} O_K} \neq 1 \forall e > 0,$$

and $i^* = \pi^{-e} u$ for $u \in O_K^*$. Weil later used this sum to show that Siegel’s main theorem on quadratic forms is a Poisson formula [7, pg. x].

The connection between the ILZF and the Generalized Exponential Sum has been known for a long time; in fact, there exists an explicit method of transforming an exponential sum into a local zeta function. This method, described in [7], requires two steps.

Let us begin with a Generalized Exponential Sum for some $f(x) \in O_K[x_1, x_2, ..., x_n]$. The first step is to use an inverse Fourier Transform to find what is known as the Local Singular Series, generally denoted $F(i)$. The meaning of such a series will not be expounded upon here, except as a go-between for our transformation; for our purposes, it will be defined in terms of the exponential sum and then summarily ignored:

$$F(i) = \int_K F^*(i^*)\psi(-i^*)di^*.$$  

The second step takes our singular series to a local zeta function using a Mellon Transform:

$$Z(s) = \int_{O_K} F(i)|i^K|^s di.$$  

Because the Igusa Local Zeta Function and the Generalized Exponential Sum are so closely linked, it would logically follow that one could find an organizing principle for the exponential sum similar to SPF for local zeta functions. As we discuss in Section 2, this is indeed the case:
Theorem 2 (SPF for Generalized Exponential Sums). Let

\[ S = \{ a \in O_k/\pi O_k | \frac{\partial f}{\partial x_i} (a) \equiv 0 \pmod{\pi} \text{ for } 1 \leq i \leq n \}. \]

Then

\[ F^*(i^*) = \sum_{a \in S} \int_{a+\pi O_k^n} \Psi(i^*f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n. \]

Although it seems that this result has never been explicitly stated in this form, it follows easily from results stated in [5] or [9].

Note that, unlike the first SPF, this organizing principle requires only the points where the partial derivatives vanish and not the points where \( f \) itself is congruent to zero; this is because the Generalized Exponential Sum does not make such a distinction. As an example, if one took \( K = \mathbb{Q}_3 \), \( i^* = 3^{-2} \), and \( f(x) = x^2 + x \), the value of \( F^*(i^*) \) comes entirely from the points where \( x \equiv 1 \pmod{3} \), even though the function itself does not vanish at these points.

This SPF for Generalized Exponential Sums gives an interesting prism through which to view the Igusa Local Zeta Function. In particular, it stands to reason that the organizing principles of the ILZF must appear either as a result of the transforms taken upon the Generalized Exponential Sum or as an organizing principle for the exponential sum itself. Since the SPF for local zeta functions examines points where \( f(x) \equiv 0 \pmod{\pi} \) and the SPF for exponential sums does not, one would assume that the need to examine such points arises from the use of the Fourier and Mellon transforms. In section 4, we find that this is indeed the case by computing zeta function directly from the transforms and proving the following theorem:

**Theorem 4.** Define

\[ N_e = \{ a \in O_k/\pi^e O_k | f(a) \equiv 0 \pmod{\pi^{e-1}} \}, \]
\[ N'_e = \{ a \in O_k/\pi^e O_k | f(a) \not\equiv 0 \pmod{\pi^e} \}, \]
\[ M(s) = \int_{O_k} |x|^s_k dx. \]

Then

\[ Z(s) = M(s) + (M(s) - q^s) \sum_{e=1}^{\infty} (q^{-es-1})(q-1)|N'_e| - |N_e|. \]

Note that if \( e = 1 \) then

\[ N_1 = \{ f(a) \not\equiv 0 \pmod{\pi} \}, \]
\[ N'_1 = \{ f(a) \equiv 0 \pmod{\pi} \}, \]

which is exactly the distinction we were missing.

The requirement of the intermediary step in the transformation from \( F^*(i^*) \) to \( Z(s) \) seems rather unwieldy; as it turns out, it is also unnecessary. In proving Theorem 4, we discover that the formulation of \( Z(s) \) in terms of \( F^*(i^*) \) can be simplified; in Section 3 of this paper, we show that the step finding \( F(i) \) can be eliminated, and \( Z(s) \) can be written explicitly in terms of \( F^*(i^*) \):

**Theorem 3.2.** Let \( M(s) \) be as above. Then

\[ Z(s) = M(s) + (M(s) - q^s) \left[ \int_K |i|^s K^{-1} F^*(i^*) di^* - M(s + 1) \right]. \]
2. Singular Points and Exponential Sums

First, we give the proof that shows that examination of points whose partial derivatives all evaluate to zero modulo $\pi$ is an organizing principle for Generalized Exponential Sums. This is essentially the same method as the one in [5].

**Theorem 2.** Let

$$F^*(i^*) = \int_{O_K^n/\pi O_K^n} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n,$$

where $i^* = \pi^{-e} u$. Further, let $\vec{a}$ be a vector such that $\vec{a} \in (O_K/\pi O_K)^n$. We define $S$ such that $\vec{a} \in S$ if $\vec{a} \equiv 0 \pmod{\pi}$ (i.e. $\vec{a}$ is a singular point mod $\pi$). If $e > 1$ then

$$F^*(i^*) = \sum_{\vec{a} \in S} \int_{\vec{a} + \pi O_K^n} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n.$$

**Proof.** We can break $F^*(i^*)$ into a sum of integrals of the vectors which are not singular points modulo $\pi$ plus a sum of integrals over the vectors which are singular points mod $\pi$ as follows:

$$F^*(i^*) = \sum_{\vec{a} \notin S} \int_{\vec{a} + \pi O_K^n} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n + \sum_{\vec{a} \in S} \int_{\vec{a} + \pi O_K^n} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n.$$  

Now, we must show that the first integral is zero. By a change of variables,

$$\int_{\vec{a} + \pi O_K^n} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n = q^{-n} \int_{O_K^n} \Psi((\pi^{-e} u)(f(a_1, a_2, ..., a_n) + \sum_{i=1}^n \pi x_i \frac{\partial f}{\partial x_i}(\vec{a}) + \pi^2(\vec{a}))) dx_1 dx_2 ... dx_n,$$

which, by Taylor’s Theorem, is equal to

$$q^{-n} \int_{O_K^n} \Psi((\pi^{-e} u)(f(a_1, a_2, ..., a_n) + \sum_{i=1}^n \pi x_i \frac{\partial f}{\partial x_i}(\vec{a}) + \pi^2(\vec{a}))) dx_1 dx_2 ... dx_n = q^{-n} \Psi((\pi^{-e} u) f(a_1, a_2, ..., a_n)) \int_{O_K^n} \Psi((\pi^{-e} u) \sum_{i=1}^n \pi x_i \frac{\partial f}{\partial x_i}(\vec{a}) + \pi^2(\vec{a}))) dx_1 dx_2 ... dx_n$$

since $\Psi$ is an additive character. By assumption, one of the partial derivatives must be congruent to a unit modulo $\pi$. So let us define $\frac{\partial f}{\partial x_i}(\vec{a}) = c_i \forall i$ and let $c_j$ be a partial which is a unit mod $\pi$ for some $j$. We change variables as follows:

$$x'_i = x_i \forall i \neq j,$$

$$x'_j = \sum_{i=1}^n c_i x_i + \pi(...)$$

A check of the Jacobian shows that this change of variables is measure invariant. Using this,
\[ \int_{O_K^*} \Psi((\pi^{-e} u) \sum_{i=1}^{n} \pi x_i \frac{\partial f}{\partial x_i}(\bar{a}) + \pi^2(...)) dx_1 dx_2 ... dx_n \]
\[ = \int_{O_K^*} \Psi((\pi^{-e+1} u) x'_1 dx'_2 ... dx'_n) \]
\[ = \int_{O_K} \Psi(\pi^{-e+1} u x'_j) dx'_j \]
\[ = 0 \]
by orthogonality relations of characters, and thus,
\[ \sum_{\bar{a} \in S} \int_{\bar{a} + \pi O_K^*} \Psi(i^* f(x_1, x_2, ..., x_n)) dx_1 dx_2 ... dx_n = 0. \]

3. A Simplified Transform from $F^*(i^*)$ to $Z(s)$

In this section, we discover a new formula which can be used to transform a Generalized Exponential Sum into an Igusa Local Zeta function. First, though, we need a lemma about the relations between roots of unity. In order to state the lemma, we must note that if $a \equiv b \pmod{O_K}$ then $\psi(a) = \psi(b)$. Thus, in dealing with $\psi$, it will often suffice to consider only representatives modulo $O_K$. In this vein, if $a \in \pi^k O_K / \pi^e O_K$ and $a' \in \pi^k O_K$ is a representative of the coset $a$, then, by abuse of notation, we will use $\psi(p^{-e} a')$ to represent the quantity $\psi(p^{-e} a)$.

Having clarified the notation, we now state the lemma:

**Lemma 3.1.** Let $u \in O_K^*$ and $k, e \in \mathbb{Z}$. If $k \leq e - 1$ then
\[ \sum_{a \in (\pi^k O_K / \pi^{k+1} O_K) / \pi^e O_K} \psi(-\pi^{-e} au) = \begin{cases} -1 & \text{if } e - 1 = k, \\ 0 & \text{if } e - 1 > k. \end{cases} \]

**Proof.** Since $a$ is a unit and $\psi$ is a non-trivial character on $\pi^{k-e} O_K$, we have
\[ \sum_{a \in \pi^k O_K / \pi^{k+1} O_K} \psi(-\pi^{-e} au) = 0 \]
by orthogonality relations of characters. If $k < e - 1$ then we also have
\[ \sum_{a \in \pi^{k+1} O_K / \pi^k O_K} \psi(-\pi^{-e} au) = 0, \]
while if $k = e - 1$ then
\[ \sum_{a \in \pi^{k+1} O_K / \pi^k O_K} \psi(-\pi^{-e} au) = \psi(0) = 1. \]
Thus,
\[
\sum_{a \in (\pi^k \mathcal{O}_K - \pi^{k+1} \mathcal{O}_K) / \pi^* \mathcal{O}_K} \psi(-\pi^{-e} au)
= \sum_{a \in \pi^* \mathcal{O}_K / \pi^* \mathcal{O}_K} \psi(-\pi^{-e} au) - \sum_{a \in \pi^{k+1} \mathcal{O}_K / \pi^* \mathcal{O}_K} \psi(-\pi^{-e} au)
= \begin{cases} 
-1 & \text{if } e - 1 = k, \\
0 & \text{if } e - 1 > k.
\end{cases}
\]

□

Now we can write an explicit formula for \( Z(s) \) in terms of \( F^*(i^*) \):

**Theorem 3.2.** Let
\[
M(s) = \int_{\mathcal{O}_K} |x|^s_K dx.
\]

Then
\[
Z(s) = M(s) + (M(s) - q^s) \left[ \int_K |i|^{s+1} F^*(i^*) d\mathcal{O}_K - M(s+1) \right].
\]

**Proof.** Recall that the zeta function can be described in terms of the Local Singular Series:
\[
Z(s) = \int_{\mathcal{O}_K} F(i)|i|^s_K di,
\]

where \( F(i) \) can in turn be expressed in terms of the Generalized Exponential Sum:
\[
F(i) = \int_K F^*(i^*) \psi(-i i^*) di^* = \int_K \int_{\mathcal{O}_K} \psi(f(x) i^*) \psi(-i i^*) dx di^*.
\]

Putting these together, we see that
\[
Z(s) = \int_{\mathcal{O}_K} \int_K \int_{\mathcal{O}_K} \psi(f(x) i^*) \psi(-i i^*) |i|^s_K d\mathcal{O}_K d\mathcal{O}_K d\mathcal{O}_K.
\]

Note that \( \mathcal{O}_K \) is of finite measure and \( K \) is sigma-finite. So we can change the order of integration. Moreover, we can break up the integral over \( K \) such that:
\[
\int_K = \int_{\mathcal{O}_K} + \sum_{e=1}^{\infty} \int_{-\mathcal{O}_K - \pi^{-e+1} \mathcal{O}_K}. \]

So \( Z(s) \) can be rewritten as
\[
\int_{\mathcal{O}_K} \int_{\mathcal{O}_K} \int_{\mathcal{O}_K} \psi(f(x) i^*) \psi(-i i^*) |i|^s_K d\mathcal{O}_K d\mathcal{O}_K d\mathcal{O}_K
+ \int_{\mathcal{O}_K} \sum_{e=1}^{\infty} \int_{-\mathcal{O}_K - \pi^{-e+1} \mathcal{O}_K} \int_{\mathcal{O}_K} \psi(f(x) i^*) \psi(-i i^*) |i|^s_K d\mathcal{O}_K d\mathcal{O}_K d\mathcal{O}_K
= \int_{\mathcal{O}_K} \int_{\mathcal{O}_K} |i|^s_K d\mathcal{O}_K d\mathcal{O}_K
+ \int_{\mathcal{O}_K} \sum_{e=1}^{\infty} \int_{-\mathcal{O}_K - \pi^{-e+1} \mathcal{O}_K} \int_{\mathcal{O}_K} \psi(f(x) i^*) \psi(-i i^*) |i|^s_K d\mathcal{O}_K d\mathcal{O}_K d\mathcal{O}_K.
\]
where the last step follows from the fact that
\[ \psi(f(x)i^*)\psi(-i^*) = 1 \forall i^* \in O_K. \]

Clearly, the first summand is \( M(s) \). So we need to show that the second summand
is equal to
\[ (M(s) - q^s) \left[ \int_K |i^*|^{s+1} F^s(i^*) di^* - M(s+1) \right]. \]

To show that this is the case, we break up the integral over \( O_K \):
\[
\int_{O_K} \sum_{k=1}^\infty \int_{\pi^{-e}O_K - \pi^{-e+1}O_K} \int_{\pi^sO_K} \psi(f(x)i^*)\psi(-i^*)|i^*_K|^{s} d\mu dx

= \int_{O_K} \sum_{k=1}^\infty \int_{\pi^{-e}O_K - \pi^{-e+1}O_K} \sum_{k=0}^{\infty} \int_{\pi^sO_K - \pi^{-e+1}O_K} \psi(f(x)i^*)\psi(-i^*)|i^*_K|^{s} d\mu dx

+ \int_{O_K} \sum_{k=1}^\infty \int_{\pi^{-e}O_K - \pi^{-e+1}O_K} \int_{\pi^sO_K} \psi(f(x)i^*)\psi(-i^*)|i^*_K|^{s} d\mu dx.
\]

For the latter triple integral, \( \frac{1}{|i^*|K} > |i^*_K|. \) So \( \psi(-i^*) = 1 \), which means that
\[
\int_{O_K} \sum_{k=1}^\infty \int_{\pi^{-e}O_K - \pi^{-e+1}O_K} \int_{\pi^sO_K} \psi(f(x)i^*)\psi(-i^*)|i^*_K|^{s} d\mu dx

= \int_{O_K} \sum_{k=1}^\infty \int_{\pi^{-e}O_K - \pi^{-e+1}O_K} q^{-e} \int_{\pi^sO_K} \psi(f(x)i^*)|\pi^e i^*_K|^{s} d\mu dx

= \int_{O_K} \sum_{k=1}^\infty \int_{\pi^{-e}O_K - \pi^{-e+1}O_K} \psi(f(x)i^*)q^{-e-s} \int_{\pi^sO_K} |i^*_K|^{s} d\mu dx

= M(s) \int_{O_K} \sum_{k=1}^\infty \int_{\pi^{-e}O_K - \pi^{-e+1}O_K} q^{-e-s} \psi(f(x)i^*) d\mu dx.
\]

where the first step is a change of variables. For the former triple integral, we note
that the innermost integral can be broken up as follows:
\[
\sum_{k=0}^{c-1} \int_{\pi^kO_K - \pi^{k+1}O_K} \psi(f(x)i^*)\psi(-i^*)|i^*_K|^{s} di

= \sum_{k=0}^{c-1} \sum_{a \in (O_K/\pi^kO_K) \times \pi^kO_K} \psi(f(x)i^*)\psi(-i^*)|i^*_K|^{s} di

= \sum_{k=0}^{c-1} \sum_{a \in (O_K/\pi^kO_K) \times \pi^kO_K} \psi(f(x)i^*)\psi(-\pi^kai^*)|\pi^kai^*_K|^{s} di.
\]

Note that all terms inside the integral are independent of \( i \). Thus, the integral is
merely the measure of \( \pi^sO_K \), which means that the above is equal to
\[
\sum_{k=0}^{c-1} \sum_{a \in (\mathbb{O}_K/\pi^{-k+1}\mathbb{O}_K)^*} \psi(f(x)i^k)\psi(-\pi^k a_i^k)|\pi^k a_i^k q^{-c} \\
= \psi(f(x)i^k) \sum_{k=0}^{c-1} \sum_{a \in (\mathbb{O}_K/\pi^{-k+1}\mathbb{O}_K)^*} \psi(-\pi^k a_i^k)q^{-k+1}q^{-c} \\
= -q^{c-(c-1)+1}\psi(f(x)i^k),
\]
where the last step is by Lemma 3.1, since \(i^k \in \pi^{-c}\mathbb{O}_K - \pi^{-c+1}\mathbb{O}_K\). So the former integral triple integral is
\[
-q^s \int_{\mathbb{O}_K}^{c} \int_{\pi^{-s-1}\mathbb{O}_K}^{\pi} \psi(f(x)i^s)|i^s|_{\mathbb{K}}^{1+s} d^s dx.
\]
Putting these together,
\[
\int_{\mathbb{O}_K}^{c} \int_{\pi^{-s-1}\mathbb{O}_K}^{\pi} \psi(f(x)i^s)\psi(-ii^s)|i^s|_{\mathbb{K}} d\pi^s d^s dx \\
= (M(s) - q^s) \int_{\mathbb{O}_K}^{c} \int_{\pi^{-s-1}\mathbb{O}_K}^{\pi} \psi(f(x)i^s)|i^s|_{\mathbb{K}}^{1+s} d^s dx.
\]
So we need only to show that the remaining double integral and sum evaluate to the bracketed expression in the lemma.

Now, note that:
\[
\int_{\mathbb{O}_K}^{c} \int_{\mathbb{O}_K}^{c} \psi(f(x)i^s)|i^s|_{\mathbb{K}}^{1+s} d^s dx = \int_{\mathbb{O}_K}^{c} \int_{\mathbb{O}_K}^{c} |i^s|_{\mathbb{K}}^{1+s} d^s dx = M(s) + 1.
\]
But
\[
\int_{\mathbb{O}_K}^{c} - \int_{\mathbb{O}_K}^{c} = \sum_{c=1}^{\infty} \int_{\pi^{-c}\mathbb{O}_K - \pi^{-c+1}\mathbb{O}_K}^{\pi} d\pi^s d^s dx.
\]
So
\[
(M(s) - q^s) \int_{\mathbb{O}_K}^{c} \sum_{c=1}^{\infty} \int_{\pi^{-c}\mathbb{O}_K - \pi^{-c+1}\mathbb{O}_K}^{\pi} \psi(f(x)i^s)|i^s|_{\mathbb{K}}^{1+s} d^s dx \\
= (M(s) - q^s) \int_{\mathbb{O}_K}^{c} \int_{\mathbb{O}_K}^{c} \psi(f(x)i^s)|i^s|_{\mathbb{K}}^{1+s} d^s dx - M(s + 1) \\
= (M(s) - q^s) \int_{\mathbb{O}_K}^{c} \int_{\mathbb{O}_K}^{c} \psi(f(x)i^s)|i^s|_{\mathbb{K}}^{1+s} dx d^s - M(s + 1) \\
= (M(s) - q^s) \int_{\mathbb{O}_K}^{c} \int_{\mathbb{O}_K}^{c} F^s(i^s)|i^s|_{\mathbb{K}}^{1+s} d^s - M(s + 1).
\]

\[\square\]

4. Transforms and Zeroes modulo \(\pi\)

In this section, we find that the need to examine the points where \(f(x)\) evaluates to zero modulo \(\pi\) is the result of the transforms which convert an Exponential Sum to a Zeta Function. To this end, we have the following theorem:
Theorem 4. Let $M(s)$ be as before, and let

$$N_e = \{a \in \mathbb{O}_k/\pi^e\mathbb{O}_K | f(a) \equiv 0 \ (mod \ \pi^{e-1}) \}, \ f(a) \neq 0 \ (mod \ \pi^e) \},$$

$$N'_e = \{a \in \mathbb{O}_k/\pi^e\mathbb{O}_K | f(a) \equiv 0 \ (mod \ \pi^e) \}.$$ 

Then

$$Z(s) = M(s) + (M(s) - q^s) \sum_{c=1}^{\infty} (q^{-c-1}-1)|q-1|N'_e - |N_e|.$$ 

Proof. From the previous section, we know that

$$\int_{\pi^{-\mathbb{O}_K}} \psi(\pi^s f(x)) dx = 0.$$ 

Splitting up the integral over $\mathbb{O}_K^n$,

$$\int_{\pi^{-\mathbb{O}_K}} \sum_{a \in (\mathbb{O}_K/\pi^e\mathbb{O}_K)^n} \psi(\pi^s f(x)) dx = 0.$$ 

$$= \int_{\pi^{-\mathbb{O}_K}} \sum_{a \in (\mathbb{O}_K/\pi^e\mathbb{O}_K)^n} q^{-en} \psi(\pi^s f(x)) dx = 0.$$ 

where the first step is a change in measure, and the second step holds because $\psi(\pi^s(..)) = 1$.

Now, we consider separately the cases when $a \in N'_e$, $a \in N_e$, and $a \notin N_e$. First, let us consider when $a \notin N_e$. So $f(a) \in \pi^e\mathbb{O}_K - \pi^{e+1}\mathbb{O}_K$, where $k < e - 1$. Then

$$\int_{\pi^{-\mathbb{O}_K}} \psi(\pi^s f(x)) dx = 0.$$ 

since, by orthogonality relations of characters,

$$\int_{\pi^{-\mathbb{O}_K}} \psi(\pi^s f(x)) dx = 0.$$ 

Next, let $a \in N_e$. So $k = e - 1$, and hence

$$\int_{\pi^{-\mathbb{O}_K}} \psi(\pi^s f(x)) dx = -q^{e-1},$$ 

since

$$\int_{\pi^{-\mathbb{O}_K}} \psi(\pi^s f(x)) dx = 0.$$ 

and

$$\int_{\pi^{e+1}\mathbb{O}_K} \psi(\pi^s f(x)) dx = q^{e-1}.$$ 

Finally, let $a \in N'_e$. Since $f(a) \in \pi^e\mathbb{O}_K$, we have $\psi = 1$, meaning that the integral is simply the measure of the area of integration:

$$\int_{\pi^{-\mathbb{O}_K}} \psi(0) dx = q^{e}(\frac{q-1}{q}).$$
We combine these to find that

\[ \sum_{a \in (O_K/\pi^n O_K)^n} q^{-en} \int_{O_K} \int_{\pi^{-n} O_K - \pi^{-n+1} O_K} \psi(i^*(f(a))) \, dx^* \, dx \]

\[ = q^{-en} \left( \sum_{a \in N_e} -q^{-e-1} \right) + \sum_{a \in N'_e} q^e \left( \frac{q - 1}{q} \right) \]

\[ = q^{-en+e-1} \left[ -|N_e| + (q - 1)|N'_e| \right], \]

from which our theorem follows. \qed

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DEPARTMENT OF MATHEMATICS, BOWDOIN COLLEGE, BRUNSWICK, ME 04011

Current address: Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218

E-mail address: wright@math.jhu.edu