1. Let \( f(x, y) = x^2 - y^2 \) with constraint function \( 2x + y = 1 \).

Using Lagrange multipliers to find all extrema.

Write \( g(x, y) = 2x + y - 1 \). Then

\[
\nabla f(x, y) = \begin{bmatrix} 2x \\ -2y \end{bmatrix}, \quad \nabla g(x, y) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

The equation \( \nabla f(x, y) = \lambda \nabla g(x, y) \) implies

\[
2x = 2\lambda, \quad -2y = \lambda, \quad 2x + y = 1.
\]

thus \( x = 2/3 \) and \( y = -1/3 \), with \( f(2/3, -1/3) = 1/3 \).

2. Consider the system of linear equations

\[
\begin{align*}
2x - y + 3z &= 3 \\
2x + y + 4z &= 4 \\
2x - 3y + 2z &= 2
\end{align*}
\]

Find the augmented matrix of the above system and use it to solve the system.

The augmented matrix is

\[
\begin{bmatrix}
2 & -1 & 3 & 3 \\
2 & 1 & 4 & 4 \\
2 & -3 & 2 & 2
\end{bmatrix}
\]

Then

\[
\begin{align*}
R_1 & \quad \begin{bmatrix} 2 & -1 & 3 & 3 \end{bmatrix} \\
R_2 & \quad \begin{bmatrix} 2 & 1 & 4 & 4 \end{bmatrix} \\
R_3 & \quad \begin{bmatrix} 2 & -3 & 2 & 2 \end{bmatrix}
\end{align*}
\begin{align*}
& \quad \left. \frac{R_1 - R_2}{R_1 - R_3} \right. \\
& \quad \left. \frac{R_5 - R_6}{R_1 - R_3} \right. \\
& \quad \left. \frac{R_6 + R_6}{R_1 - R_3} \right.
\end{align*}

\begin{align*}
R_4 & \quad \begin{bmatrix} 2 & -1 & 3 & 3 \end{bmatrix} \\
R_5 & \quad \begin{bmatrix} 0 & 2 & -1 & 1 \end{bmatrix} \\
R_6 & \quad \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
\begin{align*}
R_7 & \quad \begin{bmatrix} 2 & -1 & 3 & 3 \end{bmatrix} \\
R_8 & \quad \begin{bmatrix} 0 & 2 & -1 & 1 \end{bmatrix} \\
R_9 & \quad \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}

Therefore, \( y = (1-z)/2 \) and \( x = 7(1-z)/4 \). thus

\[
(x, y, z) \in \left\{ \left( \frac{7}{4}(1-z), \frac{1}{2}(1-z), z \right) : z \in \mathbb{R} \right\}
\]

3. Let \( f(x, y) = \sqrt{4 - x^2 - y^2} \).

(a) Find the largest possible domain and the corresponding range of \( f(x, y) \).
(b) Compute \( f_x(1, 1) \) and \( f_y(1, 1) \).
(a) The domain is \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\} \) and the range is \([0, \infty)\).

(b) \( f_x(x, y) = -x(4 - x^2 - y^2)^{-1/2} \) and \( f_y(x, y) = -y(4 - x^2 - y^2)^{-1/2} \). Hence

\[
f_x(1, 1) = -\frac{1}{\sqrt{2}} = f_y(1, 1).
\]

4. Compute

\[
\int_0^1 \ln x \, dx.
\]

Compute

\[
\int_0^1 \ln x \, dx = x \ln x \bigg|_0^1 - \int_0^1 x \frac{1}{x} \, dx = x \ln x \bigg|_0^1 - \int_0^1 dx = \left(0 - \lim_{x \to 0} x \ln x\right) - 1 = -1 - \lim_{x \to \infty} x \ln x = -1.
\]

5. Find the global extrema of

\[
f(x, y) = x^2 - 3y + y^2, \quad -1 \leq x \leq 1, \quad 0 \leq y \leq 2.
\]

The function \( f(x, y) \) has global extrema. Compute

\[
\nabla f(x, y) = \begin{bmatrix} 2x \\ -3 + 2y \end{bmatrix}, \quad \text{Hess}(f)(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}
\]

Letting \( \nabla f(x, y) = 0 \) we find that \((0.3/2)\) is in the interior of the domain of \( f \) and \( f(x, y) \) has a local minimum \(-2.25\) at \((0.3/2)\).

We now check the boundary values of \( f(x, y) \).

(i) Consider the line segment \( C_1 = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } y = 0\} \). On \( C_1 \), the function \( f \) is of the form

\[
f(x, 0) = x^2, \quad -1 \leq x \leq 1.
\]

Hence the critical point of \( f \) on \( C_1 \) is \((0, 0)\), and then \( f \) has global minimum \( 0 \) at \((0, 0)\) and has the global maximum \( 1 \) at \((-1, 0) \) and \((1, 0)\), on the line segment \( C_1 \).

(ii) Consider the line segment \( C_2 = \{(x, y) \in \mathbb{R}^2 : x = 1 \text{ and } 0 \leq y \leq 2\} \). On \( C_2 \), the function \( f \) is of the form

\[
f(2, y) = 1 - 3y + y^2, \quad 0 \leq y \leq 2.
\]

Hence, the critical point of \( f \) on \( C_2 \) is \((1, 3/2)\), and then \( f \) has the global minimum \(-1.25\) and has the global maximum \( 1 \) at \((1, 0)\), on the line segment \( C_2 \).

(iii) Consider the line segment \( C_3 = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } y = 2\} \). On \( C_3 \), the function \( f \) is of the form

\[
f(x, 2) = x^2 - 2, \quad -1 \leq x \leq 1.
\]

Hence, the critical point of \( f \) on \( C_3 \) is \((0, 2)\), and then \( f \) has the global minimum \(-2\) at \((0, 0)\) and has the global maximum \(-1 \) and \((1, 0)\) on the line segment \( C_3 \).
(iv) Consider the line segment \( C_4 = \{(x, y) \in \mathbb{R}^2 : x = -1 \text{ and } 0 \leq y \leq 2 \} \). On \( C_4 \), the function \( f \) is of the form
\[
f(-1, y) = 1 - 3y + y^2, \quad 0 \leq y \leq 2.
\]
Hence, the critical point of \( f \) on \( C_4 \) is \((-1, 3/2)\), and then \( f \) has the global minimum \(-1.25\) at \((-1, 3/2)\) and has the global maximum 1 at \((-1, 0)\).

Therefore, the function has the global maximum 1 at \((-1, 0)\) and \((1, 0)\), and the global minimum \(-2.25\) at \((0, 3/2)\).

6. Use the partial-fraction method to solve
\[
\frac{dy}{dx} = (y - 1)(y - 2)
\]
with \( y(0) = 0 \).

Compute
\[
\frac{dy}{(y - 1)(y - 2)} = dx \implies \left(\frac{1}{y - 2} - \frac{1}{y - 1}\right) dy = dx \implies \ln \left|\frac{y - 2}{y - 1}\right| = x + C_1.
\]
Thus
\[
y - 2 = C e^x \implies y = \frac{2 - C e^x}{1 - C e^x}.
\]
\( y(0) = 0 \) implies \( C = 2 \). Hence
\[
y = \frac{2 - 2e^x}{1 - 2e^x}.
\]

7. Find all candidates for local extrema and use the Hessian matrix to determine the type:
\[
f(x, y) = e^{-x^2-y^2}.
\]

Compute
\[
\nabla f(x, y) = \begin{bmatrix} -2x \\ -2y \end{bmatrix} e^{-x^2-y^2}.
\]

The only critical point is \((0, 0)\). Since
\[
\text{Hess}(f)(x, y) = \begin{bmatrix} -2 + 4x^2 & 4xy \\ 4xy & -2 + 4y^2 \end{bmatrix} e^{-x^2-y^2} \implies \text{Hess}(f)(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}
\]
the function \( f(x, y) \) has a local maximum at \((0, 0)\).

However, \( f \) is always nonpositive, \( f(x, y) \) has the global maximum at \((0, 0)\).

8. Suppose that
\[
A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}
\]
(a) Compute \( \det A \). Is \( A \) invertible?
(b) Suppose that
\[
X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]
Write \( AX = B \) as a system of linear equations.
(c) Show that if 
\[ B = \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix} \]
then \( AX = B \) has infinitely many solutions.

(a) \( \det A = 0 \). So \( A \) is not invertible.
(b) \( 2x + 4y = b_1 \) and \( 3x + 6y = b_2 \).
(c) Then the system in (b) reduces to one equation \( 2x + 4y = 3 \). Then \( AX = B \) has infinitely many solutions.

9. Solve the given initial-value problem
\[
\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\]
with \( x_1(0) = 3 \) and \( x_2(0) = -1 \).

Let
\[ A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \]
Since \( \det A = -1 \) and \( \text{tr} A = 0 \), an eigenvalue \( \lambda \) satisfies \( \lambda^2 - 1 = 0 \). So \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \).

For \( \lambda_1 = 1 \), we have
\[ \mathbf{0} = (A - \lambda_1 I_2) \mathbf{u} = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \implies \mathbf{u} = u_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \]
For \( \lambda_2 = -1 \), we have
\[ \mathbf{0} = (A - \lambda_2 I_2) \mathbf{v} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies \mathbf{v} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
Hence the general solution is
\[ \mathbf{x}(t) = c_1 e^t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
By the initial condition, we have
\[ 3c_1 + c_2 = 3, \quad 2c_1 = 4 \implies c_1 = 2, \ c_2 = -3. \]
Thus \( x_1(t) = 6e^t - 3e^{-t} \) and \( x_2(t) = 2e^t - 3e^{-t} \).

10. Suppose that
\[ \frac{dy}{dx} = y(4 - y). \]
(a) Find the equilibria of this differential equation.
(b) Compute the eigenvalues associated with each equilibrium and discuss the stability of the equilibria.

Write \( g(y) = y(4 - y) = -y^2 + 4y \). Then \( g'(y) = -2y + 4 \).
(a) The equilibria are \( y = 0 \) and \( y = 4 \).
(b) Since \( g'(0) = 4 > 0 \) and \( g'(4) = -4 < 0 \), 0 is unstable and 4 is locally stable.