Math 107 Solutions to Practice Exam 1
Part I

1. \[ \int x^3 \ln x \, dx = \ln x \frac{x^4}{4} - \int \frac{x^4}{4} \frac{1}{x} \]
   \[= \frac{x^4}{4} \ln x - \frac{x^4}{16} + C \]

2. \[\frac{1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3} \]
   \[= -\frac{1}{2} \frac{1}{x-1} + \frac{1}{2} \frac{1}{x-3} \]

Hence,
\[ \int \frac{1}{(x-1)(x-3)} \, dx = \frac{1}{2} \ln \frac{x-3}{x-1} + C \]

3. Two mg of radioactive material decays to 1.3 mg after 10 days. Write \( N(t) = 2 \cdot e^{-kt} \) Then
   \[ e^{-10k} = \frac{1.3}{2} = .65 \]
   \[-10k = \ln .65 \]
   \[k = -\ln .65 = .043 \]
   \[T_{1/2} = \frac{\ln 2}{k} = 16.12 \text{ days} \]

4. \[f'(x) = -2f(x) + 4 = -2(f(x) - 2)\]
So,

\[ f(x) - 2 = Ce^{-2t} \]

Using \( f(0) = -3 \), we find \( C = -5 \) and so,

\[ f(x) = 2 - 5e^{-2t} \]

5. To find the fourth order Taylor polynomial of \( f(x) = \sin x \) centered at \( x = \pi/2 \) we compute:

\[ f(\pi/2) = \sin \frac{\pi}{2} = 1 \]

\[ f'(\pi/2) = \cos \frac{\pi}{2} = 0 \]

\[ f''(\pi/2) \sin \frac{\pi}{2} = -1 \]

and so on. Hence

\[ P_4 = 1 - \frac{(x - \pi/2)^2}{2} + \frac{(x - \pi/2)^4}{4!} \]

Part II

6. a.

\[ e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + R_n \]

where

\[ |R_n(x)| \leq 3 \cdot \frac{x^{n+1}}{(n+1)!} \]

b. To make the error less than 0.001, we choose \( n \) so that \( (x = .1) \)
\[
3 \frac{(1)^{n+1}}{(n+1)!} < .001
\]

\[
n = 2 , \quad |R_2| < 3(.1)^3/3! = .001/2 = .0005
\]

Hence we can estimate
\[
e^1 \approx 1 + (.1) + (.1)^2/2 = 1 + .1 + .005 = 1.105
\]

(The precise answer is 1.1051709...)

7a.
\[
\int \frac{dy}{y(1-y)} = \int \left( \frac{1}{y} + \frac{1}{1-y} \right) dy
\]
\[
= \ln \left| \frac{y(t)}{1-y(t)} \right|
\]

Hence,
\[
\frac{y(t)}{1-y(t)} = Ce^{-1t}
\]

Using \( y(0) = .2 \) we find \( C = \frac{2}{8} = .25 \). Then
\[
y(t) = \frac{.25e^{-1t}}{1 + .25e^{-1t}} = 1 - \frac{1}{1 + .25e^{-1t}}
\]

b. \( y(t) = .5 \) exactly when \( e^{-1t} = .5 \) which gives the usual “half-life” time:
\[
T = \frac{\ln 2}{.1} = 6.93
\]