Problem 1 Show that following limits does not exist.

(a) \( \lim_{(x,y) \to (0,0)} \frac{x+y}{x^2+y} \)

(b) \( \lim_{(x,y) \to (0,0)} \frac{x^2+xy}{x^2+y^2} \)

Solution: (a) First, we consider the line \( y = x \) and compute the limit along this line:
\[
\lim_{x \to 0} \frac{2x}{x^2 + x} = \lim_{x \to 0} \frac{2}{x + 1} = 2
\]
On the other hand, considering the curve \( y = x^2 \) we see that
\[
\lim_{x \to 0} \frac{x + x^2}{2x^2} = \lim_{x \to 0} \frac{1 + x}{2x} = \frac{1}{2}
\]
therefore, the limit does not exist.

(b) Again computing the limit along \( y = 0 \) we get
\[
\lim_{x \to 0} \frac{x^2}{x^2} = 1.
\]
On the other hand, computing the limit along \( y = -x \) we get
\[
\lim_{x \to 0} \frac{x^2 - x^2}{2x^2} = 0
\]
hence, the limit does not exist.

Problem 2 Let \( f(x, y) = x^y \) for \( x > 0, y > 0 \). Compute \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y} \).
Solution

\[ \frac{\partial f}{\partial x} = yx^{y-1} \]
\[ \frac{\partial f}{\partial y} = \ln(x)x^y \]
\[ \frac{\partial^2 f}{\partial x^2} = (y^2 - y)x^{y-2} \]
\[ \frac{\partial^2 f}{\partial y^2} = (\ln(x))^2 x^y \]
\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{x} x^y + \ln(x)yx^{y-1} + y\ln(x)x^{y-1} \]

Problem 3 Let \( G(x, y) = x^2 + xy^2 - \frac{y^2}{2} \) and \( p = (-1, 1) \in \mathbb{R}^2 \).

(a) Calculate the equation of the plane tangent to the graph of \( G \) at the point \( p \).

(b) Calculate the gradient of \( G \) at the point \( p \) (that is, compute \( \nabla G(-1, 1) \)).

(c) Find a critical point of \( G(x, y) \) on the domain \( \mathbb{R}^2 \) and determine whether it is a local maximum, local minimum, or neither (Hint: The Hessian will help here.)

Solution

(a) The equation of the tangent plane to the graph of \( G \) at \( p \) is of the form

\[ z - z_0 = A(x - x_0) + B(y - y_0) \]

where \( A = \frac{\partial G}{\partial x}(p) \) and \( B = \frac{\partial G}{\partial y}(p) \) and \( z_0 = G(p) \). Computing these values:

\[ z_0 = G(-1, 1) = -\frac{1}{2} \]
\[ A = \frac{\partial G}{\partial x}(-1, 1) = (2x + y^2)|_{x=-1,y=1} = -1 \]
\[ B = \frac{\partial G}{\partial y}(-1, 1) = (2xy - y)|_{x=-1,y=1} = -3 \]
Thus, the equation of the plane is given by
\[ z + \frac{1}{2} = -(x + 1) - 3(y - 1). \]

(b) The Gradient of \( G \) is given by
\[ \nabla G(-1, 1) = \left[ \begin{array}{c} \frac{\partial G}{\partial x} \\ \frac{\partial G}{\partial y} \end{array} \right]_{x=-1, y=1} = \left[ \begin{array}{c} -1 \\ -3 \end{array} \right] \]

(c) To find the critical point(s) we need to solve
\[ \nabla G(x, y) = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \]
That is
\[ 2x + y^2 = 0 \]
\[ y(2x - 1) = 0 \]
The second equation implies that either \( y = 0 \) or \( 2x - 1 = 0 \). If \( y = 0 \) then \( x = 0 \) by the first equation. If \( 2x - 1 = 0 \) then \( x = \frac{1}{2} \) and from the first equation \( y^2 = -1 \) which has no real solutions. Thus the function \( G(x, y) \) has only one critical point in \( \mathbb{R}^2 \) which is \((0, 0)\).

Now, we apply the second derivative test:
\[ Hes(G)(x, y) = \left[ \begin{array}{cc} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{array} \right]_{x=0, y=0} = \left[ \begin{array}{cc} 2 & 2y \\ 2y & 2x - 1 \end{array} \right]_{x=0, y=0} = \left[ \begin{array}{cc} 2 & 0 \\ 0 & -1 \end{array} \right] \]
Since the determinant of the Hessian \( D = -2 < 0 \) by second derivative test the point \((0, 0)\) is a saddle point i.e. it is neither a local min or max.

Problem 4 \textit{Given the system}
\[ \frac{dx}{dt} = x + y \]
\[ \frac{dy}{dt} = 4x - 2y \]
(a) Solve the system for the particular solution that passes through the point \((x, y) = (1, 0)\).

(b) Find all equilibrium solutions and determine their stability.

(c) Draw the solution passing through the point \((x, y) = (1, 0)\) on the direction field for all \(t \in \mathbb{R}\). Also draw the solution passing through the point \((x, y) = (1, 1)\) for all \(t \in \mathbb{R}\). (Use the Java applet to produce the direction filed)

Solution (a) We rewrite the system in the matrix from
\[
\frac{d\vec{x}}{dt} = A\vec{x}
\]
where
\[
A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \quad \text{and} \quad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\]

Given this system, we can write the general solutions as follows: If \(A\) has two distinct real eigenvalues \(\lambda_1, \lambda_2\) and \(\vec{u}_1, \vec{u}_2\) are the corresponding eigenvectors then the general solution to the above system is given by
\[
\vec{x}(t) = c_1\vec{u}_1e^{\lambda_1 t} + c_2\vec{u}_2e^{\lambda_2 t}.
\]

To find eigenvalues we solve the characteristic equation \(det(\lambda I_2 - A) = 0\) which is \(\lambda^2 + \lambda - 6 = 0\). Thus, eigenvalues of \(A\) are \(\lambda_1 = 2\) and \(\lambda_2 = -3\). For the eigenvalue \(\lambda = 2\) the eigenvector equation \(A\vec{u} = 2\vec{u}\) leads to the system \(x + y = 2x\) and \(4x - 2y = 2y\). Remember these two equations are ALWAYS the same equation when finding the eigenvectors, and any vector \(\vec{u}\) that satisfies either equation one works. The first equation leads directly to \(x = y\). Choose \(x = 1\), so that \(y = 1\), and an eigenvector for \(\lambda_1 = 2\) is \(\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\). If we do the same thing for \(\lambda_2 = -3\), we will get the equation \(x + y = -3x\), and if we choose \(x = 1\), we get \(y = -4\), and for \(\lambda_2 = -3\), we get \(\vec{u}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}\).

Hence the general solution to this system is
\[
\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}
\]

For our particular solution, we have a starting point \(\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\). Throw that into the general solution evaluated at \(t = 0\), to get the
values of the two unknown constants \( c_1 \) and \( c_2 \) that correspond to the solution that passes through the point \((1,0)\). Hence,

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix}
\]

This leads to the two equations \( 1 = c_1 + c_2 \) and \( 0 = c_1 - 4c_2 \). Solving these two leads to \( c_1 = \frac{4}{5} \) and \( c_2 = \frac{1}{5} \). Hence our particular solution is

\[
\vec{x}(t) = \frac{4}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \frac{1}{5} \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}
\]

(b) The only equilibrium of a linear system where the matrix \( A \) has non-zero determinant (like this one) is the origin. And since the two eigenvalues here are real, distinct, non-zero, and of different signs. Thus, the origin is a saddle and unstable.

(c) On the next page, the solution passing through the point \((x,y) = (1,0)\) is drawn on the direction field for all \( t \in \mathbb{R} \). The solution passing through the point \((x,y) = (1,1)\) is also drawn. Note that these two points are marked.