Solutions

1. (a) The equilibrium solution is at \( y = 0 \). The phase line should have arrows pointing away from \( y = 0 \).
   
   (b) Your solution sketch should have a horizontal line at \( y = 0 \) (the equilibrium solution). For \( y > 0 \), the solutions should be increasing (i.e. positive slope) and for \( y < 0 \), they should be decreasing (negative slope).
   
   (c) Unstable.
   
   (d) \( y = \pm \sqrt{-\frac{1}{2} t + c} \) or \( y = 0 \).
   
   (e) \( y = \sqrt{-\frac{1}{2} t - 16} \).

2. Putting this system in matrix form we get
   
   \[
   \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
   \]
   
   To solve this we have to find the eigenvalues and eigenvectors of the matrix
   
   \[
   \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}.
   \]
   
   To find eigenvalues, we take the determinant of
   
   \[
   \begin{pmatrix} -2 - k & 1 \\ 0 & 1 - k \end{pmatrix}
   \]
   
   which is equal to
   
   \(( -2 - k)(1 - k) - 0 = (-2 - k)(1 - k) \).
   
   This is zero when \( k = 1 \) or \( k = -2 \), so these are the two eigenvalues. To find the eigenvectors we solve the equation
   
   \[
   \begin{pmatrix} -2 - k & 1 \\ 0 & 1 - k \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
   \]
   
   We get eigenvector \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) corresponding to eigenvalue \( k = -2 \) and eigenvector \( \begin{pmatrix} 1 \\ 3 \end{pmatrix} \) corresponding to \( k = 1 \). Therefore the general solution to the given system of equations is
   
   \[
   \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix}
   \]
   
   and so \( x(t) = c_1 e^{-2t} + c_2 e^t \) and \( y(t) = 3c_2 e^t \).
3. (a) The determinant of the given matrix is $2a + 4$. The matrix does not have an inverse when this is equal to zero, i.e. for $a = -2$ only.

(b) i. When $a = 1$, the matrix does have an inverse, and its inverse is
\[
\frac{1}{6} \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}
\]
So the solution is
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 10/6 \\ -1/6 \end{pmatrix}.
\]

ii. When $a = -2$, the matrix does not have an inverse. In this case, the system of equations becomes
\[
\begin{align*}
-2x + 4y &= 1 \\
-x + 2y &= -2
\end{align*}
\]
Multiplying the second equation by 2, we see that these are inconsistent. Therefore, this system of equations has no solutions.

4. (a) $\partial f/\partial x = (1-y)e^{x+y-xy}$ and $\partial f/\partial y = (1-x)e^{x+y-xy}$.

(b) $\nabla f(1,0) = (e,0)$

(c) $e/\sqrt{2}$

(d) $f(x,y) \approx e + e(x-1) + 0(y-0) = e + e(x-1)$

5. (a) The $c$-level curve is the curve where $y + x^2 = c$ or $y = -x^2 + c$. This is an inverted parabola passing through the point $(0,c)$.

(b) The $y = 1$ cross-section is a graph of $z$ against $x$ given by $z = 1 + x^2$.

(c) The gradient vector is $(2x,1)$ which at $(1,1)$ is $(2,1)$. This vector is perpendicular to the 2-level curve at the point $(1,1)$.

6. (a) The partial derivatives of $f$ are: $\partial f/\partial x = 2xy - 4y$ and $\partial f/\partial y = x^2 - 4x + 2y$. Substituting each of the three points in gives zero for both of these, so they are critical points.

(b) The Hessian for this function is
\[
Hf = \begin{pmatrix} 2y & 2x - 4 \\ 2x - 4 & 2 \end{pmatrix}.
\]
At $(0,0)$ this is
\[
\begin{pmatrix} 0 & -4 \\ -4 & 2 \end{pmatrix}
\]
which has determinant $-16$. Therefore $(0,0)$ is a saddle point.
At \((4, 0)\) the Hessian is
\[
\begin{pmatrix}
0 & 4 \\
4 & 2
\end{pmatrix}
\]
which also has determinant \(-16\). So \((4, 0)\) is also a saddle point.

At \((2, 2)\) the Hessian is
\[
\begin{pmatrix}
4 & 0 \\
0 & 2
\end{pmatrix}
\]
This has determinant \(+8\) so the critical point is either a local max or local min.
The top-left and bottom-right entries are both positive which means that \((2, 2)\) is a local minimum.

7. (a) \(P(RBWR) = 1/45, \ P(RR) = 1/15\)
(b) \(P(\text{first three socks different colors}) = 6/6 \times 4/5 \times 2/4 = 2/5\)
\(P(\text{first two socks same color}) = 6/6 \times 1/5 = 1/5.\)
(c) The answers from part (b) tell us that \(P(X = 4) = 2/5\) and \(P(X = 2) = 1/5.\)
Since the probabilities must add up to 1, this means that \(P(X = 3) = 2/5.\)
Therefore, the expectation of \(X\) is
\[
2 \times 1/5 + 3 \times 2/5 + 4 \times 2/5 = 16/5.
\]

8. (a) \(f(x) = F'(x) = \begin{cases} 2 - 2x & \text{for } 0 \leq x \leq 1; \\ 0 & \text{otherwise.} \end{cases}\)
The nonzero part of the graph is a straight line joining the points \((0, 2)\) and \((1, 0)\).
(b) \(P(0.5 \leq X \leq 1) = P(X \leq 1) - P(X \leq 0.5) = 1 - 0.5(2 - 0.5) = 1 - 0.75 = 0.25\)
(c) \(EX = \int_0^1 x(2 - 2x) \, dx = [x^2 - 2x^3/3]\bigg|_0^1 = 1/3\) and \(Var(X) = \int_0^1 (x - 1/3)^2(2 - 2x) \, dx\)
which is equal to \(\int_0^1 (-2x^3 + 10x^2/3 - 14x/9 + 2/9) \, dx = [-2x^4/4 + 10x^3/9 - 14x^2/18 + 2x/9]\bigg|_0^1 = -1/2 + 10/9 - 7/9 + 2/9 = 1/18.\)

9. (a) this is equal to \(P(X \leq 2.1) = P(Z \leq (2.1 - 2.3)/0.1) = P(Z \leq -2) = 1 - 0.9772 = 0.0228\)
(b) This is really a binomial distribution question. If \(Y\) is the number of times he breaks the record in the 16 races, then \(Y\) is binomially distributed with 16 repetitions and probability of success 0.0228. This question is then asking for \(P(Y \geq 1) = 1 - P(Y = 0)\) which is therefore \(1 - (0.9772)^{16}.\)
(c) The average time has expectation 2.3 and standard deviation
\[
\sigma = \sqrt{Var(X)/n} = 0.1/\sqrt{16} = 0.025.
\]
Therefore the probability we want is \(P(Z \leq (2.225 - 2.3)/0.025) = P(Z \leq -3) = 1 - 0.9986 = 0.0014.\)