The Homology of the
Spectrum \( bo \)
and its Connective Covers

by

Dena Sara Cowen

A dissertation submitted to The Johns Hopkins University
in conformity with the requirements for the degree of
Doctor of Philosophy

Baltimore, Maryland

1997
Abstract

In this work we recompute the known ordinary mod 2 homology of the spectrum $bo$ as a Hopf ring. In addition, we do the same for the connective covers $bo(1)$, $bo(2)$, and $bo(4)$. We compute this result using the Bar Spectral Sequence and explicit relations for the $\circ$-product and $*$-product of our elements. We also use maps from our spectrum to the Eilenberg-Maclane spectrum $K(Z/2)$ to simplify the result.
Acknowledgments

I would like to thank the faculty and staff at the Johns Hopkins University for their support, especially my advisors, Dr. W. Stephen Wilson and Dr. J. Michael Boardman, for their patience and guidance, as well as Dr. Hal Sadofsky for many hours of help and encouragement.

I would also like to thank my family and friends for their love and understanding, particularly my father for his mathematical encouragement, and my mother for her unwavering support.
Contents

0 Introduction .............................................. 1

1 Definitions and Background Materials ................. 2

1.1 Hopf rings ................................................. 2

1.2 Ω-spectra .................................................. 4

1.3 The elements $[x]$ ........................................... 5

1.4 Definition of the homotopy elements $[\alpha], [\beta]$ and $[\lambda]$ ........ 5

1.5 The elements $z_i, \overline{x}_i$ and $e$ .......................... 6

1.6 Relationships of elements in $H_*, KO_*$ ..................... 6

1.7 Types of algebras ........................................... 8

1.8 The Hopf ring for $KO$ ...................................... 8

1.9 The Frobenius and Verschiebung maps .................... 9

1.10 The bigraded algebra $\text{Tor}^R_*(\mathbb{Z}/2, \mathbb{Z}/2)$ and the suspension of $x$ 9

1.11 The bar spectral sequence ................................. 10

1.12 Maps between $H_*, bo_*$ and $H_*, H_*$ ................... 10

1.13 The notation $\alpha(i)$ ...................................... 11

2 The Computation of $H_*, bo_*$ ............................ 12

2.1 The first cycle .............................................. 12

2.2 The second cycle .......................................... 23

2.2.1 The notations $A(s)$ and $C(n, k)$ ......................... 28

2.3 The structure theorem ...................................... 34

3 The Computation of the Hopf Rings $H_*, bo\langle 4 \rangle_*$, $H_*, bo\langle 2 \rangle_*$ and $H_*, bo\langle 1 \rangle_*$ ......... 40
3.1 The computation of $H_{bo\langle 4 \rangle}$: 40
3.2 The computation of $H_{bo\langle 2 \rangle}$: 43
3.3 The computation of $H_{bo\langle 1 \rangle}$: 46
3.4 The computation of $H_{bo}$: 49

References: 50
0 Introduction

The object of this paper is to compute the Hopf ring $H_* bo_*$ for the connective real Bott spectrum $bo$. There is an obvious map of spectra to the periodic Bott spectrum, $bo \to KO$. There is also the fundamental class

$$\Theta: bo \to K(\mathbb{Z}) \to H = K(\mathbb{Z}/2).$$

We show that the sum of these maps, $\Phi: bo \to KO \times H$, induces a monomorphism for all $n$,

$$\Phi_*: H_* bo_n \to H_* KO_n \otimes H_* H_n,$$

and we compute $H_* bo_n$ as a sub-Hopf ring of the known Hopf ring on the right, where we have used the notation $H_* n = K(\mathbb{Z}/2, n)$ and $H_* X = H_* (X; \mathbb{Z}/2)$. We find that unlike most Hopf rings that have been computed, $H_* bo_*$ requires a large number of generators.

We start with some basic definitions and then proceed to the actual computation.
1 Definitions and Background Materials

1.1 Hopf rings

The following is a collection of basic facts about Hopf rings, from [1]. Let \( R \) be a graded associative commutative ring with unit. We let \( \text{CoAlg}_R \) be the category of graded cocommutative coassociative coalgebras with counit over \( R \).

Let \( H(*) = \{H_s(n)\}_{n \in \mathbb{Z}} \) be a Hopf ring over the ring \( R \). Let \( a \in H_i(n) \), \( b \in H_j(k) \), and \( c \in H_q(k) \).

Define \( \deg x \) by \( x \in H_{\deg x}(m) \). We sometimes write \( \deg x \) as \( |x| \).

(a) Each \( H_s(n) \) is an element of \( \text{CoAlg}_R \):

(i) There is a coassociative cocommutative coproduct for all \( n \),

\[ \Psi : H_s(n) \to H_s(n) \otimes H_s(n), \]

which we write as

\[ \Psi(a) = \Sigma a' \otimes a''. \]

(ii) There is a counit, \( \varepsilon : H_s(n) \to R \) such that \( \Psi \circ (1_{H_s(n)} \otimes \varepsilon) \) is the identity.

(b) Each \( H_s(k) \) is an abelian group object of \( \text{CoAlg}_R \):

(i) There is a product

\[ * : H_s(k) \otimes H_s(k) \to H_s(k) \]

which is associative and commutative.
(ii) The map \(\tau\) is in \(\text{CoAlg}_R\):

\[
\Psi(b \ast c) = \Psi(b) \ast \Psi(c) = \Sigma (b' \otimes b'') \ast (c' \otimes c''),
\]

where we use the usual \(\ast\) product on \(H_\ast(k) \otimes H_\ast(k)\), and

\[
\varepsilon(b \ast c) = \varepsilon(b)\varepsilon(c).
\]

(iii) The abelian group object unit, or zero, is \(\eta : R \to H_\ast(k)\), which is in \(\text{CoAlg}_R\). If we define \([0_k] = \eta(1) \neq 0\), then

\[
[0_k] \ast b = b.
\]

(iv) The conjugation \(\chi : H_\ast(k) \to H_\ast(k)\) has \(\chi \chi = \text{identity}\) and \(\eta \varepsilon(b) = \Sigma b' \ast \chi(b'')\). It is the abelian group object inverse.

(c) There are associative maps

\[
\circ : H_\ast(n) \otimes H_\ast(k) \to H_\ast(n + k)
\]

with the properties:

(i) The map \(\circ\) is in \(\text{CoAlg}_R\):

\[
\Psi(a \circ b) = \Psi(a) \circ \Psi(b) = \Sigma (a' \otimes a'') \circ (b' \otimes b'')
\]

and

\[
\varepsilon(a \circ b) = \varepsilon(a)\varepsilon(b).
\]

(ii) Multiplication by zero: gives

\[
[0_n] \circ b = \eta \varepsilon(b).
\]
(iii) There is a unit map $e : R \to H_*(0)$. Define

$$e(1) = [1] \in H_0(0).$$

Then $[1] \circ b = b$.

(iv) Define $\chi([1]) = [-1] \in H_0(0)$. Then

$$\chi(a) = [-1] \circ a$$

and

$$\chi(a \circ b) = \chi(a) \circ b = a \circ \chi(b).$$

(v) Commutativity:

$$a \circ b = (-1)^{ij}[-1]^n \circ b \circ a = (-1)^{ij} \chi^n(b \circ a),$$

where $a \in H_i(n)$ and $b \in H_j(k)$.

(vi) Distributivity:

$$a \circ (b \ast c) = \Sigma (-1)^{deg a'' deg b} (a' \circ b) \ast (a'' \circ c).$$

In our notation we use $a^2$ instead of $a \ast 2$ and $bc$ instead of $b \ast c$.

1.2 $\Omega$-spectra

An $\Omega$-spectrum $E$ consists of a collection of $H$-spaces $E_n$ such that for each $n \geq 0$ there is an isomorphism of $H$-spaces $E_n \simeq \Omega E_{n+1}$. Thus the spectrum $bo$ has its spaces related by $bo_n \simeq \Omega bo_{n+1}$. 
1.3 The elements \([x]\)

Let \(C^0\) be a homotopy category of topological spaces (with certain properties). Let \(E_s(\cdot)\) be an associative commutative multiplicative unreduced generalized homology theory with unit, and let \(G^*(\cdot)\) be a similar cohomology theory, both defined on \(C^0\). Let \(E_s\) and \(G^s\) denote the two coefficient rings. Let \(G^*(\cdot)\) have a representing \(\Omega\)-spectrum

\[
\underline{G}_* = \{ G_n \}_{n \in \mathbb{Z}} \in GC^0,
\]

i.e. \(G^n(X) \cong [X, G_n]\) and \(\Omega G_{n+1} \cong G_n\) (with \(GC^0\) the category of graded objects of \(C^0\)). Let \(x \in G^n\) have degree \(-n\) in the coefficient ring. Then \(x \in G^n \cong \text{[point, } G_n]\) and so we have a map in homology \(x_*: E_s \to E_s G_n\).

We define \([x] \in E_0 G_n\) to be the image of \(1 \in E_s\) under this map.

If we let \(z \in G^n\) and \(x, y \in G^k\), then in [1] we have the results:

(i) \([z] \circ [x] = [zx] = [-1]^{on} \circ [x] \circ [z]\).

(ii) \([x] * [y] = [x + y] = [y + x] = [y] * [x]\).

(iii) \(\Psi[z] = [z] \otimes [z]\).

(iv) The sub-Hopf algebra of \(E_s G_n\) generated by all \([x]\) with \(x \in G^n\) is the group ring of \(G^n\) over \(E_s\), i.e. \(E_s[G^n]\) (using (ii)).

(v) The sub-Hopf algebra of \(E_s G_*\) generated by all \([x]\) with \(x \in G^*\) is the ring ring of \(G^*\) over \(E_s\), i.e. \(E_s[G^*]\) (using (i) and (ii)).

1.4 Definition of the homotopy elements \([\alpha]\), \([\beta]\) and \([\lambda]\)

The homotopy elements \(\alpha, \beta, \lambda\) are defined by the relation:

\[
KO_* = \mathbb{Z}[\alpha, \beta, \lambda^{\pm 1}]/(\alpha^3, 2\alpha, \alpha\beta, \beta^2 - 4\lambda),
\]
where \(bo_\ast = Z[\alpha, \beta, \lambda]/(\alpha^3, 2\alpha, \alpha\beta, \beta^2 - 4\lambda) \subset KO_\ast\)

and

\[|\alpha| = 1, |\beta| = 4, |\lambda| = 8.\]

These define the elements \([\alpha], [\beta],\) and \([\lambda].\) (We have used the notation \(X_\ast = \pi_\ast X.\)

### 1.5 The elements \(z_i, \overline{z}_i\) and \(e\)

We take \(RP^\infty = 1 \times BO(1) \subset 1 \times BO \subset Z \times BO = bo_0.\) The elements \(z_i \in H_i(bo_0)\) are defined by:

\[
H_\ast RP^\infty = Z/2\{z_i : i \geq 0\},
\]

with \(z_0 = [1] \in H_\ast(Z \times BO).\) We also need the elements

\[
\overline{z}_i = z_i \ast [-1].
\]

Equivalently, \(z_i = \overline{z}_i \ast [1].\) Thus \(z_i \circ [v] = (\overline{z}_i \circ [v]) \ast [v]\) for any homotopy element \(v.\)

The fundamental class in \(H_1KO_1\) is denoted \(e\) - this is also the suspension class. The element \(1_1\) is the unit for the star product in \(H_0KO_1.\)

### 1.6 Relationships of elements in \(H_\ast KO_\ast\)

In [3] we learn of the following relationships:

(i) \(1_1 = [0_1]\)

(ii) \(e^2 = e \circ z_1\)

(iii) \((e^2)^2 = e^2 \circ z_2\)
(iv) \((e^o)^3 = 0\)

(v) \(e \circ [\alpha] = \overline{z}_1\)

(vi) \(e^{o2} \circ [\beta] = \overline{z}_2 \circ [\alpha^2]\)

(vii) \(e^{o4} \circ [\lambda] = \overline{z}_4 \circ [\beta]\)

(viii) \(\Psi(e) = e \otimes 1 + 1 \otimes e\)

(ix) \(\Psi z_k = \sum_{k=i+m} z_i \otimes z_m\)

(x) \(z_k \circ z_h = \binom{k+h}{k} z_{k+h} = \frac{(k+h)!}{k!} z_{k+h} \mod 2\)

(xi) \(\overline{z}_1 \circ \overline{z}_1 = \overline{z}_1^2\)

(xii) \(z_1 \circ [\beta] = 0\)

(xiii) \(z_2 \circ [\beta] = 0\)

(xiv) \(z_1 \circ [\alpha^2] = 0\).

We also have the equality

\[ (e^o)^n = e^o \circ z_n. \]

This only gives us a nonzero result if \(z_n\) is indecomposable, i.e. if \(n = 2^i\).

We note that \(e \circ z_i = e \circ \overline{z}_i\) for \(i > 0\), since

\[ e \circ \overline{z}_i = e \circ (z_i \ast [-1]) = (e \circ z_i) \ast (1 \circ [-1]) + (1 \circ z_i) \ast (e \circ [-1]) = e \circ z_i. \]

Also,

\[ e \circ \text{(decomposable elements)} = 0, \]

where we call an element \(x\) decomposable if \(x = ab\), with \(\varepsilon a = \varepsilon b = 0\).
1.7 Types of algebras

In this paper $E(x_1, \ldots)$ is the exterior algebra on generators $(x_1, \ldots)$, and $P(x_1, \ldots)$ is the polynomial algebra on generators $(x_1, \ldots)$.

$\Gamma(y)$ is the divided power algebra, defined as follows: it has $\mathbb{Z}/2$-module generators $\gamma_i(y)$ for $i = 0, 1, 2, \ldots$, where $|y| = n$ and

(i) $\gamma_0(y) = 1$

(ii) $\gamma_1(y) = y$

(iii) $|\gamma_k(y)| = kn$

(iv) $\gamma_k(y)\gamma_h(y) = \binom{k+h}{k} \gamma_{k+h}(y)$.

We note that as an algebra, $\Gamma(y) = E\left(\gamma_j(y) : j \geq 0\right)$. (See [2].)

1.8 The Hopf ring for $KO$

In [3] we learn that space by space $H_*KO_*$ has the following description:

$H_*KO_0 = H_*(\mathbb{Z} \times BO) = P\left(z_k : k > 0\right) \otimes P\left([1], [1]^{-1}\right)$

$H_*KO_1 = H_* (U/O) = P\left(e \circ z_{2k}\right)$

$H_*KO_2 = H_* (Sp/U) = P\left(e^{2} \circ z_{4k}\right)$

$H_*KO_3 = H_* (Sp) = E\left(e^{3} \circ z_{4k}\right)$

$H_*KO_4 = H_* (\mathbb{Z} \times BS\mathbb{P}) = P\left(z_{4k} \circ [\beta \lambda^{-1}]\right) \otimes P\left([\beta \lambda^{-1}], [\beta \lambda^{-1}]^{-1}\right)$

$H_*KO_5 = H_* (U/Sp) = E\left(e \circ z_{4k} \circ [\beta \lambda^{-1}]\right)$

$H_*KO_6 = H_* (O/U) = E\left(z_{2k} \circ [\alpha^2 \lambda^{-1}], [\alpha^2 \lambda^{-1}]^{-1}\right)$

$H_*KO_7 = H_* (O) = E\left(z_{k} \circ [\alpha \lambda^{-1}]\right)$

$H_*KO_8 = H_* (\mathbb{Z} \times BO) = P\left(z_k \circ [\lambda^{-1}] : k > 0\right) \otimes P\left([\lambda^{-1}], [\lambda^{-1}]^{-1}\right)$. 
1.9 The Frobenius and Verschiebung maps

If we are given $A$, a bicommutative Hopf algebra over $\mathbb{Z}/p$ ($p$ prime), the Frobenius map $F : A \to A$ is defined by $F(x) = x^p$. Let $A^*$ be the dual of $A$, $A^* = \text{Hom}_{\mathbb{Z}/p}(A, \mathbb{Z}/p)$. We define the Verschiebung map $V : A \to A$ as the dual of $F$ on $A^*$.

From [2] these maps have the following properties for $p = 2$:

(i) With a shift of grading $V$ and $F$ are Hopf algebra maps.

(ii) $VF = FV$:

$$V(x^{*2}) = VF(x) = FV(x) = [V(x)]^{*2}.$$  

(iii) For the coalgebra $\Gamma(x)$,

$$V\left(\gamma_{2q}(x)\right) = \gamma_q(x)$$

and

$$V\left(\gamma_q(x)\right) = 0 \text{ if } q \neq 0 \text{ mod } 2.$$  

(iv) $F(x \circ V(y)) = F(x) \circ y$.

(v) $V(x \circ y) = V(x) \circ V(y)$.

1.10 The bigraded algebra $\text{Tor}_{*,*}^R(\mathbb{Z}/2, \mathbb{Z}/2)$ and the suspension of $x$

When $R$ is an augmented $\mathbb{Z}/2$-algebra we have the following properties:

(i) $\text{Tor}_{P(x)}^R(\mathbb{Z}/2, \mathbb{Z}/2) = E[\sigma(x)]$.

(ii) $\text{Tor}_{E(x)}^R(\mathbb{Z}/2, \mathbb{Z}/2) = \Gamma[\sigma(x)]$.

Generally, $\gamma_i(\sigma(y)) \in \text{Tor}_{i,*}$ is defined whenever $y^2 = 0$. 
(iii) $\text{Tor}^{Z/2}[Z/Z, Z/2] = E[\sigma(x)]$, where $Z/2[Z] = Z/2[x, x^{-1}]$ is the group ring.

(iv) $\text{Tor}^A \otimes \text{Tor}^B \cong \text{Tor}^{A \otimes B}$.

We keep in mind the suspension of $x, \sigma(x) \in \text{Tor}_{1,*}^R(Z/2, Z/2)$, for any $x \in R$ such that $\varepsilon x = 0$. Then $\sigma(xy) = 0$ if $\varepsilon x = 0$ and $\varepsilon y = 0$, and $\sigma(z + xy) = \sigma(z)$. Also, $\sigma(x + y) = \sigma(x) + \sigma(y)$.

1.11 The bar spectral sequence

The bar spectral sequence is frequently used when we are dealing with a spectrum $E_n$ and a field $F$. It is a spectral sequence of Hopf algebras, with the basic property that:

$$E_{s,t}^2 = \text{Tor}^{H_{s+t}}(E_n; F)(F, F) \Rightarrow H_{s+t}(E_{n+1}; F),$$

provided $E_{n+1}$ is connected, with differentials

$$d_r : E_{s,t}^r \rightarrow E_{s-r, t+r-1}^r$$

For any $x \in H_*(E_n)$, $e \circ x \in H_*(E_{n+1})$ detects (the image in $E^\infty$ of) $\sigma(x)$.

1.12 Maps between $H_*bo_*$ and $H_*H_*$

We have maps

$$\Theta : H_*bo_k \rightarrow H_*K(Z, k) \rightarrow H_*H_k,$$

which will be noted in each step as needed for future use.

The algebra generators of $H_*H_*$ can be written uniquely in the form $\beta^{j_0} \circ \beta^{j_1} \circ \beta^{j_2} \circ \ldots$, where $|\beta^{(i)}| = 2^i$. We order them lexicographically,
according to the sequence of indices $(j_0, j_1, j_2, \ldots)$. Then whenever we intro-
duce a new generator of the Hopf ring $H_*$, we (generally) evaluate $\Theta$ on it,
ignoring decomposables and retaining only the earliest indecomposable term
in the ordering above. Thus

$$H_*H_k = \mathcal{E}(\beta_{(i_1)} \circ \ldots \circ \beta_{(i_k)} : i_k \geq \ldots \geq i_2 \geq i_1 \geq 0).$$

Also, $H_*H_0$ has a basis consisting of $1$ and $[1]$, with $[1]^2 = 1$, and $H_*K(\mathbb{Z}, 0) = \mathbb{Z}[z_0, z_0^{-1}]$, where $z_0 = [1]$ and $z_0^{-1} = [-1]$. We note that $\Theta$ is a morphism
of Hopf rings — it preserves $*$-products, $\circ$-products, etc. and it induces mor-
phisms of bar spectral sequences.

We note that the suspension element of $H_*H_*$ is $\beta_{(0)}$.

1.13 The notation $\alpha(i)$

The expression $\alpha(i) = k$ means that $i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$, where
$i_1 < i_2 < \ldots < i_k$. It counts the number of 1’s in the binary expansion of $i$.
Its significance is that $z_i = z_{2^{i_1}} \circ z_{2^{i_2}} \circ \ldots \circ z_{2^{i_k}}$, a $k$-fold $\circ$-product. We also
use the notation $\alpha(i) = 0$ to mean $i = 0$. 

2 The Computation of $H_{\ast}bo_{\ast}$

We now start the computation.

**STEP 0.** We start with the fact that $Z \times BO = bo_0$. By [3], $H_{\ast}(Z \times BO) = P\left(z_i, z_0^{-1} : i \geq 0\right)$. Thus

$$H_{\ast}bo_0 = P\left(z_i, z_0^{-1} : i \geq 0\right).$$

Here the map

$$\Theta : H_{\ast}bo_0 \to H_{\ast}K(Z, 0) \to H_{\ast}H_0$$

is clearly given by

$$\Theta(z_0) = [1], \Theta(z_0^{-1}) = [-1] = [1], \text{ and } \Theta(z_i) = 0$$

for $i > 0$.

### 2.1 The first cycle

In steps 1-8, we show that multiplication by $[\lambda]$,

$$\circ [\lambda] : H_{\ast}bo_n \to H_{\ast}bo_{n-8} = H_{\ast}KO_{n-8} \simeq H_{\ast}KO_n$$

is a monomorphism.

In steps 1, 2, 4 and 8, we find it useful to search for the lowest 0 digit in the binary expansion of $i$, by writing

$$i = 1 + 2 + \cdots + 2^{m-1} + 2^{m+1}q = 2^m(2q + 1) - 1,$$

where $m \geq 0$ and $q \geq 0$.

**STEP 1.** Now we use the bar spectral sequence to find $H_{\ast}bo_1$. Since

$$H_{\ast}bo_0 = P\left(z_i, z_0^{-1} : i \geq 0\right) = P\left(z_i : i \geq 1\right) \otimes P\left(z_0, z_0^{-1}\right),$$
we have

\[ E_{s,t}^2 = \text{Tor}_{s,t}^H(b_0) (\mathbb{Z}/2, \mathbb{Z}/2) = E \left( \sigma(z_i) : i \geq 0 \right) \Rightarrow H_* b_0. \]

The suspension \( \sigma(x) \) lies in the first filtration of the bar spectral sequence. Therefore the generators of the \( E^2 \)-term are all in \( E^2_{1,*} \), collapsing the bar spectral sequence. This gives \( E \left( \sigma(z_i) : i \geq 0 \right) \) in the \( E^\infty \)-term.

The element \( e \circ z_i \in H_* b_0 \) detects \( \sigma(z_i) \). To determine the algebra structure of \( H_* b_0 \), we start from the fact that \( e^2 = e \circ z_1 \), and use \( V(z_{2i}) = z_i \), together with the fact that \( F \left( V(x) \circ y \right) = x \circ F(y) \). Thus

\[(e \circ z_i)^2 = F(e \circ z_i) = F(e) \circ z_{2i} = e^2 \circ z_{2i} = (e \circ z_1) \circ z_{2i} = e \circ z_{2i+1}.\]

We apply this as often as possible to \( e \circ z_i \) by writing \( i = 2^m (2q + 1) - 1 \) as above; then \( e \circ z_i = F^m(e \circ z_{2q}) \).

Therefore

\[ H_* b_0 = P \left( e \circ z_{2i} : i \geq 0 \right). \]

For the new element \( e \), we clearly have \( \Theta(e) = \beta(0) \).

**STEP 2.** To find \( H_* b_0 \): Since \( H_* b_0 = P \left( e \circ z_{2i} : i \geq 0 \right) \), the bar spectral sequence is

\[ E_{s,t}^2 = \text{Tor}_{s,t}^H(b_0) (\mathbb{Z}/2, \mathbb{Z}/2) = E \left( \sigma(e \circ z_{2i}) : i \geq 0 \right) \Rightarrow H_* b_0. \]

Once again the suspension \( \sigma(x) \) lies in the first filtration, collapsing the bar spectral sequence at the \( E^2 \)-term. This gives \( E \left( \sigma(e \circ z_{2i}) : i \geq 0 \right) \) in the \( E^\infty \)-term.

The element \( e^{o2} \circ z_{2i} \in H_* b_0 \) detects \( \sigma(e \circ z_{2i}) \). To determine the algebra structure of \( H_* b_0 \), we start from the fact that \( (e^{o2})^2 = e^{o2} \circ z_2 \), and use
$V(z_{4i}) = z_{2i}$. Thus

$$(e^{o^2} \circ z_{2i})^2 = F(e^{o^2} \circ z_{4i}) = F(e^{o^2} \circ z_{2i} = (e^{o^2})^2 \circ z_{4i} = (e^{o^2} \circ z_{2i}) \circ z_{4i} = e^{o^2} \circ z_{4i+2}.$$  

We apply this as often as possible to $e^{o^2} \circ z_{2i}$ by writing $i = 2^m(2q + 1) - 1$; then $e^{o^2} \circ z_{2i} = F^{m}(e^{o^2} \circ z_{4q})$.

We therefore have

$$H_*b_2 = P(e^{o^2} \circ z_{4i} : i \geq 0).$$  

**STEP 3.** To find $H_*b_3$: Since $H_*b_2 = P(e^{o^2} \circ z_{4i} : i \geq 0)$, the bar spectral sequence is

$$E_{s,t}^2 = \text{Tor}^{H_*b_2}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = E\left(\sigma(e^{o^2} \circ z_{4i}) : i \geq 0\right) \Rightarrow H_*b_3.$$  

The suspension $\sigma(e^{o^2} \circ z_{4i})$ collapses the bar spectral sequence at the $E^2$-term. Therefore, in the $E^\infty$-term, we have $E\left(\sigma(e^{o^2} \circ z_{4i}) : i \geq 0\right)$.

As usual, $e^{o^3} \circ z_{4i} \in H_*b_3$ detects $\sigma(e^{o^2} \circ z_{4i})$. Since $(e^{o^3})^2 = 0$ in $H_*b_3$,

$$(e^{o^3} \circ z_{4i})^2 = F(e^{o^3} \circ z_{4i}) = F(e^{o^3}) \circ z_{8i} = (e^{o^3})^2 \circ z_{8i} = 0.$$  

Thus

$$H_*b_3 = E\left(e^{o^3} \circ z_{4i} : i \geq 0\right).$$  

**STEP 4.** To find $H_*b_4$: Since $H_*b_3 = E\left(e^{o^3} \circ z_{4i} : i \geq 0\right)$, the bar spectral sequence is

$$E_{s,t}^2 = \text{Tor}^{H_*b_3}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = E\left(\sigma(e^{o^3} \circ z_{4i}) : i \geq 0\right) \Rightarrow H_*b_4.$$  

Once again the suspension $\sigma(x)$ lies in the first filtration, but the elements $\gamma_j\left(\sigma(e^{o^3} \circ z_{4i})\right)$ are in the $j$-th filtration. Since all of these elements are in even total degree $s + t$, the bar spectral sequence collapses at the $E^2$-term.
By the definition of a divided power algebra, the generators are the $\gamma_{2i}$, so we have
\[ E^\infty = \Gamma\left( \sigma\left( e^{o^3} \circ z_{4i} \right) : i \geq 0 \right), \]
which can be rewritten as
\[ E\left( \gamma_{2i}(\sigma( e^{o^3} \circ z_{4i}) ; i, j \geq 0 \right). \]

To determine the algebra structure of $H_*\mathfrak{bo}_4$, we start from the fact that
\[(e^{o^4})^2 = e^{o^4} \circ z_{4i}, \]
and use $V(z_{8i}) = z_{8i}$. As usual, $e^{o^4} \circ z_{4i}$ detects $\sigma( e^{o^3} \circ z_{4i})$, giving
\[ F(e^{o^4} \circ z_{4i}) = F(e^{o^4}) \circ z_{8i} = (e^{o^4} \circ z_{4}) \circ z_{8i} = e^{o^4} \circ z_{8i+4}. \]

We apply this as often as possible to $e^{o^4} \circ z_{4i}$ by writing $i = 2^m(2q + 1) - 1$; then $e^{o^4} \circ z_{4i} = F^m(e^{o^4} \circ z_{8q}).$

Suppose $x \in H_*\mathfrak{bo}_4$ detects $\gamma_{2i}\left( \sigma (e^{o^3} \circ z_{8q}) \right)$. Since
\[ V^j\left( \gamma_{2i}(\sigma( e^{o^3} \circ z_{8q})) \right) = \sigma(e^{o^3} \circ z_{8q}), \]
we must have $V^j x = e^{o^4} \circ z_{8q}$. Further,
\[ V^j F^m x = F^m(e^{o^4} \circ z_{8q}) = e^{o^4} \circ z_{4i} \]
detects $\sigma( e^{o^3} \circ z_{4i})$ and $F^m x$ detects $\gamma_{2j}(\sigma(e^{o^3} \circ z_{4i}))$. Thus the elements $x$ (as $j$ and $q$ vary) are polynomial generators of $H_*\mathfrak{bo}_4$.

To identify $x$, we consider the image of $H_*\mathfrak{bo}_4$ under $o[\lambda]$ in the known algebra $H_*\mathfrak{bo}_{-4} = H_*\mathfrak{ko}_{-4} = P\left( z_{4i} \circ [\beta] : i > 0 \right) \otimes P\left( [\beta], [\beta]^{-1} \right)$. We make use of the Hopf ring properties that
\[ (i) \ e^{o^4} \circ [\lambda] = z_4 \circ [\beta] \]
\[ (ii) \ z_{2k+1} \circ [\beta] = z_{2k} \circ z_1 \circ [\beta] = 0 \]
(iii) \( z_{4k+2} \circ [\beta] = z_{4k} \circ z_2 \circ [\beta] = 0. \)

Then

\[
V^j(x \circ [\lambda]) = e^{o_4} \circ z_{8q} \circ [\lambda] = z_4 \circ z_{8q} \circ [\beta] = z_{8q+4} \circ [\beta] + \text{decomposables}.
\]

Since \( V(\tau_{2k}) = \tau_k \) and \( V(\tau_{2k+1}) = 0 \), we conclude that

\[
x \circ [\lambda] = z_{2i} (8q+4) \circ [\beta] + \ldots
\]

We write

\[
x = z_{2i} (8q+4) \circ [\beta \lambda^{-1}] + \ldots \in H_4 KO_4,
\]

so that \( F^m \left( z_{2i} (8q+4) \circ [\beta \lambda^{-1}] \right) + \ldots \) detects \( \gamma_{2i} \left( \sigma (e^{o_3} \circ z_{4i}) \right) \).

We know that \( \tau_{2k+1} \circ [\beta] = 0 \) and \( \tau_{4k+2} \circ [\beta] = 0 \). Since any positive integer divisible by 4 has the form \( 2^i (8q+4) \) uniquely, the image of \( H_4 KO_4 \) is the whole polynomial ring \( P \left( \tau_{4i} \circ [\beta] : i > 0 \right) \).

Thus we write formally

\[
H_4 KO_4 = P \left( \tau_{4i} \circ [\beta \lambda^{-1}] : \alpha(i) \geq 1 \right) \subset H_4 KO_4.
\]

Note that \( [\beta \lambda^{-1}] \) itself is not an element of \( H_4 KO_4 \).

We clearly have \( \Theta(\tau_4 \circ [\beta \lambda^{-1}]) = \Theta(e^{o_4}) = \beta_{(0)}^{o_4} \). Since \( V^i (\tau_{2i+2}) = \tau_4 \) and \( V^i (\beta_{(i)}) = \beta_{(i)} \), we deduce

\[
\Theta(\tau_{2i+2} \circ [\beta \lambda^{-1}]) = \beta_{(i)}^{o_4} + \ldots
\]

where the unstated terms involve \( \beta_{(j)} \) with \( j < i \).

**STEP 5.** To find \( H_5 BO_5 \): Since \( H_5 BO_5 = P \left( \tau_{4i} \circ [\beta \lambda^{-1}] : \alpha(i) \geq 1 \right) \), the bar spectral sequence is

\[
E_{si}^2 = \text{Tor}_{s,i}^{H_5 BO_5} (\mathbb{Z}/2, \mathbb{Z}/2) = E \left( \sigma (\tau_{4i} \circ [\beta \lambda^{-1}]) : \alpha(i) \geq 1 \right) \Rightarrow H_5 BO_5.
\]
Once again the suspension $\sigma(x)$ lies in the first filtration, which collapses the bar spectral sequence at the $E^2$-term.

Then $e \circ z_{4i} \circ [\beta \lambda^{-1}] = e \circ \varphi_{4i} \circ [\beta \lambda^{-1}]$ detects $\sigma(\varphi_{4i} \circ [\beta \lambda^{-1}])$. We have

$$F(e \circ z_{4i} \circ [\beta]) = F(e) \circ z_{8i} \circ [\beta] = (e \circ z_1) \circ z_{8i} \circ [\beta] = 0,$$

since $z_1 \circ [\beta] = 0$.

Therefore

$$H_*^{\text{bo}_5} = E \left( e \circ z_{4i} \circ [\beta \lambda^{-1}] : \alpha(i) \geq 1 \right).$$

Note that there is an injection from $H_*^{\text{bo}_5}$ to

$$H_*^{\text{bo}_{-3}} = H_*^{\text{KO}_{-3}} = E \left( e \circ z_{4k} \circ [\beta] : k \geq 0 \right).$$

We also have

$$\Theta(e \circ \varphi_{2i+2} \circ [\beta \lambda^{-1}]) = \beta_{(0)} \circ \beta_{(i)}^4 + \ldots$$

**STEP 6.** To find $H_*^{\text{bo}_6}$: Since $H_*^{\text{bo}_5} = E \left( e \circ z_{4i} \circ [\beta \lambda^{-1}] : \alpha(i) \geq 1 \right)$, the bar spectral sequence is

$$E^2_{n,t} = \text{Tor}^{H_*^{\text{bo}_5}}(\mathbb{Z}/2, \mathbb{Z}/2) = \Gamma \left( \sigma(e \circ z_{4i} \circ [\beta \lambda^{-1}]) : \alpha(i) \geq 1 \right) \Rightarrow H_*^{\text{bo}_6}.$$}

The suspension $\sigma(x)$ lies in the first filtration, but the elements $\gamma_j(x)$ are in the $j$-th filtration. Since all elements have even total degree, the bar spectral sequence collapses at the $E^2$-term. Thus

$$E^\infty = \Gamma \left( \sigma(e \circ z_{4i} \circ [\beta \lambda^{-1}]) : \alpha(i) \geq 1 \right),$$

which can be rewritten as

$$E \left( \gamma_2(\sigma(e \circ z_{4i} \circ [\beta \lambda^{-1}])) : \alpha(i) \geq 1, j \geq 0 \right).$$
As usual, $e^{o^2} \circ \tau_{4i} \circ [\beta \lambda^{-1}] = e^{o^2} \circ z_{4i} \circ [\beta \lambda^{-1}] \in H^*\mathbb{L}_6$ detects $\sigma(e \circ z_{4i} \circ [\beta \lambda^{-1}])$. We use the Hopf ring fact that $e^{o^2} \circ [\beta] = \tau_2 \circ [\alpha^2]$ to rewrite this as
\[
\tau_2 \circ \tau_{4i} \circ [\alpha^2 \lambda^{-1}] = \tau_{4i+2} \circ [\alpha^2 \lambda^{-1}] + \ldots
\]
Again we apply $o[\lambda]$ to map $H^*\mathbb{L}_6$ into the known algebra
\[
H^*\mathbb{L}_{-2} = H^*\mathbb{K} = E\left(\tau_{2i} \circ [\alpha^2] : i > 0\right) \otimes E\left([\alpha^2] - 1\right).
\]
We know that $\tau_{2i+1} \circ [\alpha^2] = 0$, so no odd $\tau$'s appear. Suppose $x \in H^*\mathbb{L}_6$ detects $\gamma_{2i}\left(\sigma(e \circ z_{4i} \circ [\beta \lambda^{-1}])\right)$. Then
\[
V^jx = e^{o^2} \circ \tau_{4i} \circ [\beta \lambda^{-1}] = \tau_{4i+2} \circ [\alpha^2 \lambda^{-1}] + \ldots
\]
and
\[
V^j(x \circ [\lambda]) = \tau_{4i+2} \circ [\alpha^2] + \ldots
\]
Thus, we must have $x \circ [\lambda] = \tau_{2i}(4i+2) \circ [\alpha^2] + \ldots$

Thus $H^*\mathbb{L}_6 \to H^*\mathbb{L}_{-2}$ is monic and we can identify $H^*\mathbb{L}_6$ with its image in $H^*\mathbb{L}_{-2}$. Every number $2k$ with $\alpha(k) \geq 2$ can be written uniquely in the form $2^j(4i+2)$ with $i > 0$. Hence
\[
H^*\mathbb{L}_6 = E\left(\tau_{2i} \circ [\alpha^2 \lambda^{-1}] + \ldots : \alpha(i) \geq 2\right),
\]
where $\tau_{2i}(4i+2) \circ [\alpha^2 \lambda^{-1}] + \ldots$ detects $\gamma_{2i}\left(\sigma(e \circ z_{4i} \circ [\beta \lambda^{-1}])\right)$.

Clearly,
\[
\Theta(\tau_2 \circ \tau_{2i+2} \circ [\alpha^2 \lambda^{-1}]) = \Theta(e^{o^2} \circ \tau_{2i+2} \circ [\beta \lambda^{-1}]) = \beta^{o^2}_{(0)} \circ \beta^{o^4}_{(i)} + \ldots
\]
Since $\Theta$ commutes with $V^j$ and $\tau_2 \circ \tau_{2i+2} = \tau_{2+2i+2} + \ldots$, we deduce
\[
\Theta\left(\tau_{2j+1+2i+j+2} \circ [\alpha^2 \lambda^{-1}]\right) = \beta^{o^2}_{(j)} \circ \beta^{o^4}_{(i+j)} + \ldots
\]
We rewrite this as
\[ \Theta(\overline{\varepsilon}_{2;1+1+2;2+2} \circ [\alpha^2 \lambda^{-1}]) = \beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 4} + \ldots \]
for \( i_2 \geq i_1 \geq 0 \).

**STEP 7.** To find \( H_7 \mathbf{b} \mathbf{o}_7 \): Since \( H_7 \mathbf{b} \mathbf{o}_6 = E(\overline{\varepsilon}_{2i} \circ [\alpha^2 \lambda^{-1}] + \ldots : \alpha(i) \geq 2) \),
the bar spectral sequence is
\[ E^2_{s,t} = \text{Tor}_{s,t}^H(\mathbb{Z}/2, \mathbb{Z}/2) = \Gamma \left( \sigma(\overline{\varepsilon}_{2i} \circ [\alpha^2 \lambda^{-1}]) : \alpha(i) \geq 2 \right) \Rightarrow H_7 \mathbf{b} \mathbf{o}_7. \]

We rewrite
\[ \Gamma \left( \sigma(\overline{\varepsilon}_{2i} \circ [\alpha^2 \lambda^{-1}]) : \alpha(i) \geq 2 \right) \]
as
\[ E \left( \gamma_{2i}(\sigma(\overline{\varepsilon}_{2i} \circ [\alpha^2 \lambda^{-1}]))) : \alpha(i) \geq 2, j \geq 0 \right). \]

The suspension \( \sigma(x) \) lies in the first filtration, but the elements \( \gamma_{2i}(x) \) are in the \( 2^j \)-th filtration. Thus, for the moment, we gain no information about the behavior of the bar spectral sequence.

As usual, \( e \circ \overline{\varepsilon}_{2i} \circ [\alpha^2 \lambda^{-1}] \) detects \( \sigma(\overline{\varepsilon}_{2i} \circ [\alpha^2 \lambda^{-1}]) \). We now use the Hopf ring fact that \( e \circ [\alpha] = \overline{\varepsilon}_1 \) to write this as
\[ \overline{\varepsilon}_1 \circ \overline{\varepsilon}_{2i} \circ [\alpha \lambda^{-1}] = \overline{\varepsilon}_{2i+1} \circ [\alpha \lambda^{-1}] + \ldots \]
Again we apply \( \circ [\lambda] \) to map \( H_7 \mathbf{b} \mathbf{o}_7 \) into the known algebra
\[ H_7 \mathbf{b} \mathbf{o}_{-1} = H_7 \mathbf{KQ}_{-1} = E \left( \overline{\varepsilon}_i \circ [\alpha] : i > 0 \right) \otimes E \left( [\alpha] - 1 \right). \]

Since \( \mathbf{b} \mathbf{o}_{-1} \) maps as a subset into a spectral sequence that collapses, the bar spectral sequence for \( \mathbf{b} \mathbf{o}_7 \) collapses.

Suppose that \( x \in H_7 \mathbf{b} \mathbf{o}_7 \) detects \( \gamma_{2i}(\sigma(\overline{\varepsilon}_{2i} \circ [\alpha^2 \lambda^{-1}])) \). Then
\[ V^j x = e \circ \overline{\varepsilon}_{2i} \circ [\alpha^2 \lambda^{-1}] = \overline{\varepsilon}_1 \circ \overline{\varepsilon}_{2i} \circ [\alpha \lambda^{-1}] = \overline{\varepsilon}_{2i+1} \circ [\alpha \lambda^{-1}] + \ldots \]
and
\[ V^j(x \circ [\lambda]) = \bar{z}_{2i+1} \circ [\alpha] + \ldots \]

We must have \( x \circ [\lambda] = \bar{z}_{2i+(2i+1)} \circ [\alpha] + \ldots \), where \( \alpha(i) \geq 2 \). Every number \( k \) with \( \alpha(k) \geq 3 \) can be written uniquely in the form \( 2^j(2i + 1) \) with \( \alpha(i) \geq 2 \).

Thus \( H_*b_{07} \to H_*b_{08} \) is monic and we write
\[
H_*b_{07} = E\left( \bar{z}_i \circ [\alpha \lambda^{-1}] + \ldots : \alpha(i) \geq 3 \right).
\]

Clearly,
\[
\Theta\left( \bar{z}_1 \circ \bar{z}_{21+1+2^2+2} \circ [\alpha \lambda^{-1}] \right) = \Theta\left( \bar{z} \circ \bar{z}_{21+1+2^2+2} \circ [\alpha \lambda^{-1}] \right) = \beta_{(0)} \circ \beta_{(i_1)} \circ \beta_{(i_2)} + \ldots
\]

for \( i_2 \geq i_1 \). Since \( \Theta \) commutes with \( V^j \), we deduce
\[
\Theta\left( \bar{z}_{2i+2^j+i_1+1+2^i+i_2+2} \circ [\alpha \lambda^{-1}] \right) = \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} + \ldots
\]

For any \( i \) with \( \alpha(i) = 3 \), we therefore have, after reindexing,
\[
\Theta\left( \bar{z}_i \circ [\alpha \lambda^{-1}] \right) = \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} + \ldots
\]

where \( i = 2i_3 + 2i_2 + 2^i + 2^i \) is the binary expansion of \( i \) with \( i_3 \geq i_2 \geq i_1 \).

**STEP 8.** To find \( H_*b_{08} \): Since \( H_*b_{07} = E\left( \bar{z}_i \circ [\alpha \lambda^{-1}] + \ldots : \alpha(i) \geq 3 \right) \),
the bar spectral sequence is
\[
E^2_{s,t} = \text{Tor}_{s,t}^H b_{07} (\mathbb{Z}/2, \mathbb{Z}/2) = \Gamma\left( \sigma(\bar{z}_i \circ [\alpha \lambda^{-1}]) : \alpha(i) \geq 3 \right) \Rightarrow H_*b_{08}.
\]

We note that
\[
\Gamma\left( \sigma(\bar{z}_i \circ [\alpha \lambda^{-1}]) : \alpha(i) \geq 3 \right)
\]

can be rewritten as
\[
E^2\left( \gamma_{2^j}(\sigma(\bar{z}_i \circ [\alpha \lambda^{-1}])) : \alpha(i) \geq 3, j \geq 0 \right).
\]
As usual, \( \sigma(\tau_i \circ [\alpha \lambda^{-1}]) \) is detected by \( e \circ \tau_i \circ [\alpha \lambda^{-1}] \). Since \( e \circ [\alpha] = \tau_1 \), we can rewrite this as \( \tau_1 \circ \tau_i \circ [\lambda^{-1}] \).

We wish to simplify \( \tau_1 \circ \tau_i \circ [\lambda^{-1}] \) by using the fact that

\[
\tau_1 \circ \tau_{2i} = \tau_{2i+1} + \text{decomposables}.
\]

Then

\[
\tau_1 \circ \tau_{2i+1} = \tau_1 \circ \tau_1 \circ \tau_{2i} + \tau_1 \circ \text{decomposables}.
\]

Since \( \Psi(\tau_1) = \tau_1 \otimes 1 + 1 \otimes \tau_1 \), we have \( \tau_1 \circ \text{decomposables} = 0 \). Hence

\[
\tau_1 \circ \tau_{2i+1} \circ [\lambda^{-1}] = \tau_1 \circ \tau_1 \circ \tau_{2i} \circ [\lambda^{-1}] = F(\tau_1) \circ \tau_{2i} \circ [\lambda^{-1}] = F(\tau_1 \circ \int \circ [\lambda^{-1}]).
\]

We apply this relation as often as possible, by expanding \( i \) in binary form and looking for the lowest 0 digit: \( i = 2^m(2q + 1) - 1 \), where \( q, m \geq 0 \). Then

\[
\tau_1 \circ \tau_i \circ [\lambda^{-1}] = F^m(\tau_1 \circ \tau_{2q} \circ [\lambda^{-1}]) = F^m(\tau_{2q+1} \circ [\lambda^{-1}]) + \ldots
\]

Here \( m + \alpha(2q + 1) = m + \alpha(q) + 1 = \alpha(i) + 1 \geq 4 \).

To see that the bar spectral sequence collapses, apply \( \circ [\lambda] \) to map to the known bar spectral sequence for \( H_* BO \), which does collapse. As before, \( \gamma_2 \left( \sigma(\tau_i \circ [\alpha \lambda^{-1}]) \right) \) is detected by

\[
F^m \left( e \circ \tau_i \circ [\alpha \lambda^{-1}] \right) + \ldots = F^m \left( \tau_{2q(2q+1)} \circ [\lambda^{-1}] \right) + \ldots
\]

Therefore,

\[
H_*BO_8 = P \left( \tau_i \circ [\lambda^{-1}] + \ldots : \alpha(i) \geq 4 \right) \otimes P \left( F^j(\tau_i \circ [\lambda^{-1}]) + \ldots : \alpha(i) + j = 4, i, j \geq 1 \right).
\]

We need the value of \( \Theta \) on \( F^m(\tau_k \circ [\lambda^{-1}]) + \ldots \) whenever \( \alpha(k) + m = 4 \) (including the case \( m = 0 \)). As above, we write \( k = 2^j(2q + 1) \), so that
$$\alpha(k) = \alpha(q) + 1. \text{ Then}$$

$$V^j \left( F^n(z_k \circ [\lambda^{-1}]) \right) = F^n(z_{2^m+1} \circ [\lambda^{-1}]) = z_1 \circ z_i \circ [\lambda^{-1}] = e \circ z_i \circ [\alpha \lambda^{-1}],$$

where

$$i = 2^m(2q + 1) - 1 = (2^m - 1) + 2^{m+1}q,$$

so that $\alpha(i) = m + \alpha(q) = m + \alpha(k) - 1 = 3$. Thus we let

$$i = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2}$$

be the binary expansion of $i$, where $i_3 \geq i_2 \geq i_1$. We know

$$\Theta(e \circ z_i \circ [\alpha \lambda^{-1}]) = \beta_{(0)} \circ \beta_{(i_1)} \circ \beta^{o2}_{(i_2)} \circ \beta^{o4}_{(i_3)} + \ldots$$

Since $\Theta$ commutes with $V$, we must have

$$\Theta \left( F^n(z_k \circ [\lambda^{-1}]) \right) = \beta_{(0)} \circ \beta_{(i_1 + i_2)} \circ \beta^{o2}_{(i_2)} \circ \beta^{o4}_{(i_3)} + \ldots$$

We reindex as

$$\Theta \left( F^j(z_i \circ [\lambda^{-1}]) \right) = \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta^{o2}_{(i_3)} \circ \beta^{o4}_{(i_4)} + \ldots$$

whenever $\alpha(i) + j = 4$. Here, we write $i = 2^i(2q + 1)$, where $2^i$ is the largest power of 2 that divides $i$, and use the binary expansion

$$2^{2i} - 2^{i_1} = (2^{2i_1} - 2^{i_1}) + 2^{i_1+1}q = 2^{i_2} + 2^{i_3+1} + 2^{i_4+2},$$

where $i_4 \geq i_3 \geq i_2 \geq i_1$. We note that $i_2 > i_1$ if $j = 0$, but that $i_2 = i_1$ if $j > 0$. 

2.2 The second cycle

From step 9 on, \( \circ [\lambda] \) will no longer be a monomorphism. Instead, we treat \( H_s \mathfrak{b}_n \) as a sub-Hopf algebra of \( H_s \mathfrak{b}_{n-8} \otimes H_s \mathcal{H}_n \subset H_s KO_{n-8} \otimes H_s \mathcal{H}_n \) by means of \( \circ [\lambda] \) and \( \Theta \). To check that the bar spectral sequence for \( H_s \mathfrak{b}_n \) collapses, it is only necessary to verify that the map \( \mathfrak{b}_{n-1} \to KO_{n-1} \times \mathcal{H}_{n-1} \) induces a monomorphism on the \( E^2 \)-terms, thus embedding the spectral sequence in one that is known to collapse. This will be clear in practice. We make frequent appeal to the structure of \( H_s \mathfrak{b}_{n-8} \). Each factor of \( H_s \mathfrak{b}_{n-1} \) will lead to one or more factors of \( H_s \mathfrak{b}_n \).

**STEP 9.** To find \( H_s \mathfrak{b}_{9} \): Since \( H_s \mathfrak{b}_{8} \) is known, the bar spectral sequence is

\[
E^2_{s,t} = \text{Tor}^{H_s \mathfrak{b}_{8}}(\mathbb{Z}/2, \mathbb{Z}/2) = E \left( \sigma(\overline{z}_i \circ [\lambda^{-1}]) : \alpha(i) \geq 4 \right) \otimes E \left( \sigma(F^j(\overline{z}_i \circ [\lambda^{-1}])) : \alpha(i) + j = 4, i, j \geq 1 \right)
\]

\( \Rightarrow H_s \mathfrak{b}_{9} \).

This is the last step in which we will explicitly state the bar spectral sequence.

As usual, \( e \circ z_i \circ [\lambda^{-1}] + \ldots = e \circ \overline{z}_i \circ [\lambda^{-1}] + \ldots \) detects \( \sigma(\overline{z}_i \circ [\lambda^{-1}]) \). We use the relation \( e \circ z_{2q+1} = F(e \circ z_i) \) as often as possible, as in step 1. Write \( i = 2^m(2q + 1) - 1 \) where \( \alpha(i) = m + \alpha(q) \geq 4 \); then

\[
e \circ z_i \circ [\lambda^{-1}] + \ldots = F^m(e \circ z_{2q} \circ [\lambda^{-1}]) + \ldots
\]

detects \( \sigma(\overline{z}_i \circ [\lambda^{-1}]) \). Thus if \( \alpha(q) \geq 4 \), the element \( e \circ z_{2q} \circ [\lambda^{-1}] + \ldots \) is a polynomial generator of \( H_s \mathfrak{b}_{9} \). But if \( \alpha(q) < 4 \) the polynomial generator is \( F^m(e \circ z_{2q} \circ [\lambda^{-1}]) + \ldots \) instead, with \( m = 4 - \alpha(q) \). This takes care of the first exterior algebra in \( E^2 = E^\infty \).
The method of applying \(\circ[\lambda]\) fails for the second exterior algebra. Although \(F^j(\overline{z}_i \circ [\lambda^{-1}]) + \ldots\) is a generator of \(H_9\mathfrak{b}o_9\), it becomes decomposable in \(H_9\mathfrak{b}o_{9,0}\)
and \(\circ[\lambda]\) annihilates \(F^j(\overline{z}_i \circ [\lambda^{-1}])\) at the \(E^2\)-level. Instead, we apply the morphism \(\Theta\) of bar spectral sequences to the bar spectral sequence for \(H_9\mathfrak{H}_9\). Suppose \(x \in H_9\mathfrak{b}o_9\) is the element that detects \(F^j(\overline{z}_i \circ [\lambda^{-1}])\); then \(x \circ [\lambda] = 0\), and by step 8, \(\Theta(x)\) detects \(\sigma(\beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{g2} \circ \beta_{(i_4)}^{g4} + \ldots)\) and therefore is

\[
\Theta(x) = \beta_{(0)} \circ \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{g2} \circ \beta_{(i_4)}^{g4} + \ldots
\]

Also, since \(j > 0\), we have \(i_2 = i_1\) here. Hence \(\circ[\lambda]\) and \(\Theta\) define a monomorphism

\[
H_9\mathfrak{b}o_9 \to H_9KQ_9 \otimes H_9\mathfrak{H}_9 = H_9(KQ_9 \times \mathfrak{H}_9)
\]

which makes it clear that \(x^2 = 0\), as \(H_9\mathfrak{H}_9\) is an exterior algebra. We therefore treat \(H_9\mathfrak{b}o_9\) as a subalgebra of \(H_9KQ_9 \otimes H_9\mathfrak{H}_9\), and label those generators that map trivially to \(H_9KQ_9\) by their images under \(\Theta\) instead. Thus, we write

\[
H_9\mathfrak{b}o_9 = P\left(e \circ z_2i \circ [\lambda^{-1}] : \alpha(i) \geq 4\right)
\]
\[
\otimes P\left(F^j(e \circ z_2i \circ [\lambda^{-1}]) + \ldots : \alpha(i) + j = 4, j \geq 1\right)
\]
\[
\otimes E\left(\beta_{(i_1)} \circ \beta_{(i_2)}^{g2} \circ \beta_{(i_3)}^{g2} \circ \beta_{(i_4)}^{g4} + \ldots : i_4 \geq i_3 \geq i_2 \geq i_1 = 0\right).
\]

The generators of the third factor are already defined by their images under \(\Theta\). We need the images of the generators of the form \(F^j(e \circ z_2i \circ [\lambda^{-1}]) + \ldots\) with \(\alpha(i) + j = 4\) (including the case \(j = 0\)). We again use \(F^j(e \circ z_2i) = e \circ z_k\), where \(k = 2^j(2i + 1) - 1\), and note that \(\alpha(k) = j + \alpha(i) = 4\). By step 8,

\[
\Theta\left(F^j(e \circ z_2i \circ [\lambda^{-1}]) + \ldots\right) = \Theta(e \circ z_k \circ [\lambda^{-1}] + \ldots)
\]
\[
= \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{g2} \circ \beta_{(i_4)}^{g4} + \ldots
\]

with indices defined by the binary expansion \(k = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2}\),
with $i_5 \geq i_4 \geq i_3 > i_2 \geq i_1 = 0$. Since $i_3 > i_2$, this is different from all the generators in the third factor of $H_\ast b_9$.

**STEP 10.** To find $H_\ast b_{10}$: We use $H_\ast b_9$.

The first factor in $E^2$ is

$$E\left(\sigma(e \circ z_{2i} \circ [\lambda^{-1}]) : \alpha(i) \geq 4\right),$$

where $e^{o_2} \circ z_{2i} \circ [\lambda^{-1}]$ detects $\sigma(e \circ z_{2i} \circ [\lambda^{-1}])$. As in step 2 we write $i = 2^m(2q + 1) - 1$ where $\alpha(i) = m + \alpha(q) \geq 4$; then

$$e^{o_2} \circ z_{2i} \circ [\lambda^{-1}] = F^m(e^{o_2} \circ z_{4q} \circ [\lambda^{-1}])$$

detects $\sigma(e \circ z_{2i} \circ [\lambda^{-1}])$. As in step 9 (with indices doubled), we deduce polynomial generators $e^{o_2} \circ z_{4q} \circ [\lambda^{-1}]$ (with $\alpha(q) \geq 4$) and $F^m(e^{o_2} \circ z_{4q} \circ [\lambda^{-1}])$ (with $\alpha(q) + m = 4, m \geq 1$) in $H_\ast b_{10}$.

The second factor in $E^2$ is

$$E\left(\sigma(F^j(e \circ z_{2i} \circ [\lambda^{-1}])) : \alpha(i) + j = 4, j \geq 1\right),$$

which is annihilated by $\circ [\lambda]$, so we therefore apply $\Theta$ instead. By step 9,

$$\Theta\left(\sigma(e \circ z_{k} \circ [\lambda^{-1}]) + \ldots\right) = \Theta\left(\sigma(F^j(e \circ z_{2i} \circ [\lambda^{-1}]))\right)$$

$$= \sigma(\beta(0) \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{o_2} \circ \beta_{(i_5)}^{o_4} + \ldots)$$

where we use the binary expansion

$$k = 2^j(2i + 1) - 1 = 2^{i_3} + 2^{i_4+1} + 2^{i_5+2}$$

with $i_5 \geq i_4 \geq i_3 > i_2$. Since $j > 0$, $k$ is odd and $i_2 = 0$. This element is therefore detected by $\beta_{(i_2)}^{o_3} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{o_2} \circ \beta_{(i_5)}^{o_4} + \ldots$

We rewrite the third factor

$$\Gamma\left(\sigma(\beta_{(i_1)} \circ \beta_{(i_2)}^{o_2} \circ \beta_{(i_3)}^{o_2} \circ \beta_{(i_4)}^{o_4}) : i_4 \geq i_3 \geq i_2 \geq i_1 = 0\right)$$
in $E^2$ as

$$E\left(\gamma_{2j}(\sigma(\beta_{(0)} \circ \beta_{(i_2)}^{o2} \circ \beta_{(i_3)}^{o2} \circ \beta_{(i_4)}^{o4})) : i_4 \geq i_3 \geq i_2 \geq 0, j \geq 0\right).$$

The generator shown is thus detected by

$$\beta_{(j)} \circ \beta_{(j)} \circ \beta_{(j+i_2)}^{o2} \circ \beta_{(j+i_3)}^{o2} \circ \beta_{(j+i_4)}^{o4} + \ldots$$

Generally, a factor

$$E(\beta_{(j_1)} \circ \ldots \circ \beta_{(j_m)})$$

in $H_s \mathbb{b}_n$ gives a factor

$$\Gamma\left(\sigma(\beta_{(j_1)} \circ \ldots \circ \beta_{(j_m)})\right)$$

in $E^2$. This yields the factor

$$E\left(\beta_{(j)} \circ \beta_{(j+i)} \circ \ldots \circ \beta_{(j+i_m)} : j \geq 0\right)$$

in $H_s \mathbb{b}_{n+1}$, which we reindex. In future, we use this without comment.

Therefore,

$$H_s \mathbb{b}_{10} = P\left(e^{o2} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\right)$$

$$\otimes P\left(F^j(e^{o2} \circ z_{4i} \circ [\lambda^{-1}]) + \ldots : \alpha(i) + j = 4, j \geq 1\right)$$

$$\otimes E\left(\beta_{(i_1)}^{o3} \circ \beta_{(i_2)}^{o2} \circ \beta_{(i_3)}^{o2} \circ \beta_{(i_4)}^{o4} : i_4 \geq i_3 \geq i_2 > i_1 = 0\right)$$

$$\otimes E\left(\beta_{(i_1)}^{o2} \circ \beta_{(i_2)}^{o2} \circ \beta_{(i_3)}^{o2} \circ \beta_{(i_4)}^{o4} : i_4 \geq i_3 \geq i_2 \geq i_1 \geq 0\right),$$

and we again have a monomorphism

$$H_s \mathbb{b}_{10} \rightarrow H_s KO_2 \otimes H_s H_{10} = H_s(KO_{10} \times H_{10})$$

We need to compute $\Theta\left(F^j(e^{o2} \circ z_{4i} \circ [\lambda^{-1}])\right)$ whenever $\alpha(i) + j = 4$ and $i, j \geq 0$. Since

$$F^j(e^{o2} \circ z_{4i} \circ [\lambda^{-1}]) = e^{o2} \circ z_{2k} \circ [\lambda^{-1}] = e \circ (e \circ z_{2k} \circ [\lambda^{-1}]),$$
where $k = 2^j(2i + 1) - 1$ (thus giving $\alpha(k) = j + \alpha(i) = 4$), step 9 gives

$$\Theta \left( F^j(e^{ov} \circ z_{4i} \circ [\lambda^{-1}]) \right) = \beta^{ov}_{(i_5)} \circ \beta_{(i_4)} \circ \beta^{ov}_{(i_3)} \circ \beta^{ov}_{(i_2)} + \ldots ,$$

where we use the binary expansion $2k = 2i_5 + 2i_4 + 2i_3 + 2i_2 + 2i_1$ with indices $i_5 \geq i_4 \geq i_3 > i_2 > i_1 = 0$. Again, we observe that the conditions on the indices ensure that this element differs from all the previously mentioned generators of $H_{sbo_{10}}$. (In the future, we suppress any mention of this point.)

**STEP 11.** To find $H_{sbo_{11}}$: We use $H_{sbo_{10}}$.

The factor $P\left( e^{ov} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4 \right)$ in $H_{sbo_{10}}$ leads to

$$E\left( e^{ov} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4 \right),$$

just as in step 3.

The factor

$$P\left( F^j(e^{ov} \circ z_{4i} \circ [\lambda^{-1}]) + \ldots : \alpha(i) + j = 4, j \geq 1 \right)$$

gives rise to the factor

$$E\left( \sigma(F^j(e^{ov} \circ z_{4i} \circ [\lambda^{-1}])) : \alpha(i) + j = 4, j \geq 1 \right)$$

in $E^2$. However, $F^j(e^{ov} \circ z_{4i} \circ [\lambda^{-1}])$ decomposes in $H_{sbo_2}$ and we must apply $\Theta$ instead. By step 10, $\sigma(F^j(e^{ov} \circ z_{4i} \circ [\lambda^{-1}]))$ is detected by

$$\beta_{(0)} \circ (\beta^{ov}_{(0)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta^{ov}_{(i_4)} \circ \beta^{ov}_{(i_5)}),$$

where $i_5 \geq i_4 \geq i_3 > i_2$. Since $j > 0$, we have $i_2 = 1$.

The two exterior factors are handled as usual.
Thus
\[ H_{ba_{11}} = E \left( e^{\alpha(i)} \circ \left[ \lambda^{-1} \right] : \alpha(i) \geq 4 \right) \]
\[ \otimes E \left( \beta_{(i_1)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 4} + \cdots : i_5 \geq i_4 \geq i_3 > i_2 = 1, i_1 = 0 \right) \]
\[ \otimes E \left( \beta_{(i_1)}^{\circ 4} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \cdots : i_4 \geq i_3 \geq i_2 > i_1 \geq 0 \right) \]
\[ \otimes E \left( \beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \cdots : i_5 \geq i_4 \geq i_3 \geq i_2 \geq i_1 \geq 0 \right). \]

We need to evaluate \( \Theta(e^{\alpha(i)} \circ z_{4i} \circ [\lambda^{-1}]) \) when \( \alpha(i) = 4 \). By factoring
\( e^{\alpha(i)} \circ z_{4i} \circ [\lambda^{-1}] = e^{\alpha(i)} \circ (z_{4i} \circ [\lambda^{-1}]) \), we see from step 8 that
\[ \Theta(e^{\alpha(i)} \circ z_{4i} \circ [\lambda^{-1}]) = e^{\beta_{(i_1)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 4} + \cdots} \]
where we use the binary expansion \( 4i = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2} \), with
\( i_5 \geq i_4 \geq i_3 > i_2 \geq 2 \).

2.2.1 The notations \( A(s) \) and \( C(n, k) \)

Now we define some notation for the basic families of elements. We define
\[ A(s) = \text{the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)} \text{, for } s \geq 1, \]
\[ A(0) = [1] \]
and inductively
\[ C(n, k) = \text{the set of all } \beta_{(i_k)} \circ \beta_{(i_{k+1})} \circ \beta_{(i_{k+2})}^{\circ 2} \circ \beta_{(i_{k+3})}^{\circ 4} \circ C(n - 1, k + 4), \]
where \( i_{k+1} > i_k \geq i_{k-1} + 3 \), starting from \( C(0, k) = [1] \). We also assume implicitly that \( i_m \geq i_n \geq 0 \) whenever \( m > n \).
Thus we rewrite

\[ H_{* b_{11}} = E\left(e_{i}^{3} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4 \right) \]

\[ \otimes E\left(\beta_{(i_{1})}^{3} \circ \beta_{(i_{2})} \circ \beta_{(i_{3})} \circ \beta_{(i_{4})}^{4} \circ \beta_{(i_{5})}^{4} + \ldots : i_{3} > i_{2} = 1, i_{1} = 0 \right) \]

\[ \otimes E\left(\beta_{(i_{1})}^{4} \circ \beta_{(i_{2})} \circ \beta_{(i_{3})}^{2} \circ \beta_{(i_{4})}^{4} + \ldots : i_{2} > i_{1} \right) \]

\[ \otimes E\left( A(1) \circ \beta_{(i_{2})}^{2} \circ \beta_{(i_{2})}^{2} \circ \beta_{(i_{4})}^{2} \circ \beta_{(i_{5})}^{4} + \ldots \right) . \]

**STEP 12.** To find \( H_{* b_{12}} \): We use \( H_{* b_{11}} \). Only the first factor of \( H_{* b_{11}} \) requires any discussion. It gives rise to the factor

\[ \Gamma \left( \sigma(e_{i}^{3} \circ z_{4i} \circ [\lambda^{-1}]) : \alpha(i) \geq 4 \right) \]

in the \( E^{2} \)-term, which we rewrite as

\[ E\left( \gamma_{2j} \left( \sigma(e_{i}^{3} \circ z_{4i} \circ [\lambda^{-1}]) \right) : \alpha(i) \geq 4, j \geq 0 \right) . \]

By step 4, \( \gamma_{2j} \left( \sigma(e_{i}^{3} \circ z_{4i} \circ [\lambda^{-1}]) \right) \) is detected by the element written formally as \( F^{m}\left( z_{2(8q+4)} \circ [\beta \lambda^{-2}] \right) + \ldots \in H_{* b_{12}} \), where as before we write

\[ i = 2^{m}(2q + 1) - 1, \]

so that \( \alpha(i) = m + \alpha(q) \geq 4. \)

We thus obtain

\[ P\left( z_{4i} \circ [\beta \lambda^{-2}] : \alpha(i) \geq 5 \right) \otimes P\left( F^{j}(z_{4i} \circ [\beta \lambda^{-2}]) : \alpha(i) + j = 5, i, j \geq 1 \right) . \]

Therefore

\[ H_{* b_{12}} = P\left( z_{4i} \circ [\beta \lambda^{-2}] : \alpha(i) \geq 5 \right) \]

\[ \otimes P\left( F^{j}(z_{4i} \circ [\beta \lambda^{-2}]) + \ldots : \alpha(i) + j = 5, i, j \geq 1 \right) \]

\[ \otimes E\left( \beta_{(i_{1})}^{4} \circ \beta_{(i_{2})} \circ \beta_{(i_{3})}^{2} \circ \beta_{(i_{4})}^{4} + \ldots : i_{3} > i_{2} = i_{1} + 1 \right) \]

\[ \otimes E\left( A(1) \circ \beta_{(i_{2})}^{2} \circ \beta_{(i_{3})}^{4} \circ \beta_{(i_{4})}^{2} \circ \beta_{(i_{5})}^{4} + \ldots \right) \]

\[ \otimes E\left( A(2) \circ \beta_{(i_{2})}^{2} \circ \beta_{(i_{3})}^{2} \circ \beta_{(i_{4})}^{2} \circ \beta_{(i_{5})}^{4} + \ldots \right) . \]
From step 11,
\[
\Theta \left( e^{\alpha_4} \circ z_{4i} \circ [\lambda^{-1}] \right) = \beta_{(0)}^{\alpha_4} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\alpha_2} \circ \beta_{(i_5)}^{\alpha_4}
\]
whenever \( \alpha(i) = 4 \), where we use the binary expansion
\[
4i = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2},
\]
so that \( i_3 > i_2 \geq 2 \). By step 4, we can rewrite
\[
e^{\alpha_4} \circ z_{4i} \circ [\lambda^{-1}] = F^m (\pi_{8q+4} \circ [\beta \lambda^{-2}]) + \ldots
\]
where \( i = 2^m (2q + 1) - 1 \), so that \( \alpha(8q + 4) + m = \alpha(q) + 1 + m = \alpha(i) + 1 = 5 \).

Since \( \Theta \) commutes with \( V^j \), we deduce that
\[
\Theta \left( F^m \left( \pi_{2^j(8q+4)} \circ [\beta \lambda^{-2}] \right) + \ldots \right) = \beta_{(i_1)}^{\alpha_4} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\alpha_2} \circ \beta_{(i_5)}^{\alpha_4} + \ldots
\]

We reindex this as
\[
\Theta \left( F^j \left( \pi_{4i} \circ [\beta \lambda^{-2}] \right) + \ldots \right) = \beta_{(i_1)}^{\alpha_4} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\alpha_2} \circ \beta_{(i_5)}^{\alpha_4} + \ldots
\]
where \( 2^{i_1} \) is the largest power of 2 that divides \( i \), which we have set equal to \( 2^{i_1} (2q + 1) \), as above, and we use the binary expansion
\[
2^j \cdot 4i - 2^{i_1+2} = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2},
\]
where \( i_3 > i_2 \geq i_1 + 2 \). We note that \( i_2 = i_1 + 2 \) if \( j > 0 \), and that \( i_2 \geq i_1 + 3 \) if \( j = 0 \).

**Step 13.** To find \( H_s b_0 \cdot 13 \): We use \( H_s b_0 \cdot 12 \).

By step 5, the factor
\[
P \left( \pi_{4i} \circ [\beta \lambda^{-2}] + \ldots \mid \alpha(i) \geq 5 \right)
\]
in \( H_s b_2 \), yields the factor
\[
E \left( e \circ z_{4i} \circ [\beta \lambda^{-2}] : \alpha(i) \geq 5 \right)
\]
in \( H_s b_{12} \).

The factor
\[
P \left( F^j(z_{4i} \circ [\beta \lambda^{-2}]) + \ldots : \alpha(i) + j = 5, i, j \geq 1 \right)
\]
in \( H_s b_{12} \) yields the factor
\[
E \left( \sigma (F^j(z_{4i} \circ [\beta \lambda^{-2}])) : \alpha(i) + j = 5, i, j \geq 1 \right)
\]
in the \( E^2 \)-term. As in step 11, the generators map trivially to \( H_s b_5 \) and we must apply \( \Theta \) instead. By step 12, the generator shown is detected by
\[
\beta(0) \circ \beta^{04}_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta^{02}_{(i_4)} \circ \beta^{04}_{(i_5)} \circ \beta^{04}_{(i_6)},
\]
where \( i_3 > i_2 = i_1 + 2 \) (since \( j > 0 \)).

Thus,
\[
H_s b_{13} = E \left( e \circ z_{4i} \circ [\beta \lambda^{-2}] : \alpha(i) \geq 5 \right)
\]
\[
\otimes E \left( \beta_{(i_1)} \circ \beta^{04}_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{02}_{(i_5)} \circ \beta^{04}_{(i_6)} + \ldots : i_4 > i_3 = i_2 + 2, i_1 = 0 \right)
\]
\[
\otimes E \left( A(1) \circ \beta^{04}_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{02}_{(i_5)} \circ \beta^{04}_{(i_6)} + \ldots : i_4 > i_3 = i_2 + 1 \right)
\]
\[
\otimes E \left( A(2) \circ \beta^{04}_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{02}_{(i_5)} \circ \beta^{04}_{(i_6)} + \ldots : i_4 > i_3 \right)
\]
\[
\otimes E \left( A(3) \circ \beta^{02}_{(i_4)} \circ \beta^{02}_{(i_5)} \circ \beta^{02}_{(i_6)} \circ \beta^{04}_{(i_7)} + \ldots \right).
\]

We need to know \( \Theta(e \circ z_{4i} \circ [\beta \lambda^{-2}]) \) whenever \( \alpha(i) = 5 \). By decomposing within \( H_s b_{10} \), we see from steps 5 and 8 that
\[
\Theta(e \circ z_{4i} \circ [\beta \lambda^{-2}]) = \left( \beta(0) \circ \beta^{04}_{(i_2)} \right) \circ \left( \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{02}_{(i_5)} \circ \beta^{04}_{(i_6)} \right) + \ldots
\]
where we use the binary expansion

\[ A_i = 2^{i_2 + 2} + 2^{i_3} + 2^{i_4} + 2^{i_5 + 1} + 2^{i_6 + 2}, \]

so that \( i_4 > i_3 \geq i_2 + 3 \).

**STEP 14.** Now we can use everything we learned in step 6 to find \( H_{b_0 b_{14}} \).

Thus

\[
H_{b_0 b_{14}} = E \left( \frac{\alpha^2 \lambda^{-2}}{\beta(i)} : \alpha(i) \geq 6 \right) \\
\quad \times E \left( \beta^{\circ 2}_{(i_1)} \circ \beta^{\circ 4}_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{\circ 2}_{(i_5)} \circ \beta^{\circ 4}_{(i_6)} + \ldots : i_4 > i_3 = i_2 + 2 \right) \\
\quad \times E \left( A(2) \circ \beta^{\circ 4}_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)} \circ \beta^{\circ 2}_{(i_6)} \circ \beta^{\circ 4}_{(i_7)} + \ldots : i_5 > i_4 = i_3 + 1 \right) \\
\quad \times E \left( A(3) \circ \beta^{\circ 4}_{(i_4)} \circ \beta_{(i_5)} \circ \beta^{\circ 2}_{(i_6)} \circ \beta^{\circ 4}_{(i_7)} + \ldots : i_6 > i_5 \right) \\
\quad \times E \left( A(4) \circ \beta^{\circ 2}_{(i_5)} \circ \beta^{\circ 2}_{(i_6)} \circ \beta^{\circ 2}_{(i_7)} \circ \beta^{\circ 4}_{(i_8)} + \ldots \right). 
\]

We need to know \( \Theta(\beta^{\circ 2}_{(i_1)} \circ \beta^{\circ 4}_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{\circ 2}_{(i_5)} \circ \beta^{\circ 4}_{(i_6)} + \ldots) \), whenever \( \alpha(i) = 6 \). Again we can factor within \( H_{b_0 b_{14}} \) and read off from step 6 and 8 that

\[
\Theta(\beta^{\circ 2}_{(i_1)} \circ \beta^{\circ 4}_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{\circ 2}_{(i_5)} \circ \beta^{\circ 4}_{(i_6)} + \ldots) = \beta^{\circ 2}_{(i_1)} \circ \beta^{\circ 4}_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{\circ 2}_{(i_5)} \circ \beta^{\circ 4}_{(i_6)} + \ldots,
\]

where we use the binary expansion

\[ 2i = 2^{i_1 + 1} + 2^{i_2 + 2} + 2^{i_3} + 2^{i_4} + 2^{i_5 + 1} + 2^{i_6 + 2}, \]

so that \( i_4 > i_3 \geq i_2 + 3 \).

**STEP 15.** Now we use everything we learned in step 7 to find \( H_{b_0 b_{15}} \).
Thus
\[
H_{b_{15}} = E\left(\bar{z}_i \circ [\alpha \lambda^{-2}] : \alpha(i) \geq 7\right)
\]
\[
\otimes E\left(A(1) \circ \beta^{(i_2)} \circ \beta^{(i_3)} \circ \beta^{(i_4)} \circ \beta^{(i_5)} \circ \beta^{(i_6)} \circ \beta^{(i_7)} + \ldots: i_5 > i_4 = i_3 + 2\right)
\]
\[
\otimes E\left(A(3) \circ \beta^{(i_4)} \circ \beta^{(i_5)} \circ \beta^{(i_6)} \circ \beta^{(i_7)} \circ \beta^{(i_8)} + \ldots: i_6 > i_5 = i_4 + 1\right)
\]
\[
\otimes E\left(A(4) \circ \beta^{(i_4)} \circ \beta^{(i_5)} \circ \beta^{(i_6)} \circ \beta^{(i_7)} + \ldots: i_6 > i_5\right)
\]
\[
\otimes E\left(A(5) \circ \beta^{(i_5)} \circ \beta^{(i_6)} \circ \beta^{(i_7)} \circ \beta^{(i_8)} + \ldots\right).
\]

For \(\alpha(i) = 7\) (by using steps 7 and 8), we have
\[
\Theta(\bar{z}_i \circ [\alpha \lambda^{-2}]) = \beta^{(i_1)} \circ \beta^{(i_2)} \circ \beta^{(i_3)} \circ \beta^{(i_4)} \circ \beta^{(i_5)} \circ \beta^{(i_6)} \circ \beta^{(i_7)} + \ldots,
\]
where we use the binary expansion
\[
i = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2} + 2^{i_4} + 2^{i_5} + 2^{i_6+1} + 2^{i_7+2},
\]
so that \(i_5 > i_4 \geq i_3 + 3\).
2.3 The structure theorem

The general pattern should now be apparent.

**Theorem 2.3.1** The Hopf ring $H_\ast \mathbb{bo}_{\ast}$ is a sub-Hopf ring of $H_\ast (KO_{\ast} \times H_\ast)$ and is the (graded) tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of $H_\ast KO_{\ast}$:

   \[ P\left( z_i \circ [\lambda^{-n}] + \ldots : \alpha(i) \geq 4n \right), \text{on } \mathbb{bo}_{8n} \]
   \[ P\left( e \circ z_0 \circ [\lambda^{-n}] \right), \text{on } \mathbb{bo}_{8n}, \text{ for } n \leq 0 \]
   \[ P\left( e^{3^2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n \right), \text{on } \mathbb{bo}_{8n+2} \]
   \[ E\left( e^{3^3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n \right), \text{on } \mathbb{bo}_{8n+3} \]
   \[ P\left( e^{3^4} \circ [\beta \lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 1 \right), \text{on } \mathbb{bo}_{8n+4} \]
   \[ P\left( e \circ z_0 \circ [\beta \lambda^{-n}], z_0^{-1} \circ [\beta \lambda^{-n}] \right), \text{on } \mathbb{bo}_{8n-4}, \text{ for } n \leq 0 \]
   \[ E\left( e \circ z_0 \circ [\beta \lambda^{-n+1}] : \alpha(i) \geq 4n + 1 \right), \text{on } \mathbb{bo}_{8n+5} \]
   \[ E\left( z_{2i} \circ [\alpha^2 \lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 2 \right), \text{on } \mathbb{bo}_{8n+6} \]
   \[ E\left( [\alpha^2 \lambda^{-n}] - 1 \right), \text{on } \mathbb{bo}_{8n-2}, \text{ for } n \leq 0 \]
   \[ E\left( z_i \circ [\alpha \lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 3 \right), \text{on } \mathbb{bo}_{8n+7} \]
   \[ E\left( [\alpha \lambda^{-n}] - 1 \right), \text{on } \mathbb{bo}_{8n-1}, \text{ for } n \leq 0. \]
2. Polynomial algebras on generators that decompose in $H_{\text{KQ}}$, companions to the polynomial algebras in the first family:

$$P\left(F^j(z_i \circ \lfloor \lambda^{-n} \rfloor) + \ldots : \alpha(i) + j = 4n, i, j \geq 1\right), \text{on } \mathfrak{bo}_{8n}$$

$$P\left(F^j(e \circ z_{2i} \circ \lfloor \lambda^{-n} \rfloor) + \ldots : \alpha(i) + j = 4n, j \geq 1\right), \text{on } \mathfrak{bo}_{8n+1}$$

$$P\left(F^j(\alpha^2 \circ z_{4i} \circ \lfloor \lambda^{-n} \rfloor) + \ldots : \alpha(i) + j = 4n, j \geq 1\right), \text{on } \mathfrak{bo}_{8n+2}$$

$$P\left(F^j(\alpha^3 \circ z_{6i} \circ \lfloor \lambda^{-n} \rfloor) + \ldots : \alpha(i) + j = 4n + 1, i, j \geq 1\right), \text{on } \mathfrak{bo}_{8n+4}.$$

3. Exterior algebras involving $\beta(0)$ that arise from the second family:

$$E\left(\beta(0) \circ \beta^2_{(i_2)} \circ \beta^2_{(i_3)} \circ \beta^4_{(i_4)} \circ C(n, 5) + \ldots\right), \text{on } \mathfrak{bo}_{8n+9}$$

$$E\left(\beta^3_{(i_3)} \circ \beta_{(i_2)} \circ \beta^2_{(i_3)} \circ \beta^4_{(i_4)} \circ C(n, 5) + \ldots : i_2 > 0\right), \text{on } \mathfrak{bo}_{8n+10}$$

$$E\left(\beta^3_{(i_3)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta^2_{(i_4)} \circ \beta^4_{(i_5)} \circ C(n, 6) + \ldots : i_3 > i_2 = 1\right), \text{on } \mathfrak{bo}_{8n+11}$$

$$E\left(\beta(0) \circ \beta^4_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^2_{(i_5)} \circ \beta^4_{(i_6)} \circ C(n, 7) + \ldots : i_4 > i_3 = i_2 + 2\right), \text{on } \mathfrak{bo}_{8n+13}.$$

4. General exterior algebras that arise from the third family by unlimited suspension:

$$E\left(A(s) \circ \beta^2_{(i+1)} \circ \beta^2_{(i+2)} \circ \beta^2_{(i+3)} \circ \beta^4_{(i+4)} \circ C(n, s + 5) + \ldots\right)$$

$$E\left(A(s) \circ \beta^4_{(i+1)} \circ \beta_{(i+2)} \circ \beta^2_{(i+3)} \circ \beta^4_{(i+4)} \circ C(n, s + 5) + \ldots : i_{s+2} > i_{s+1}\right)$$

$$E\left(A(s) \circ \beta^4_{(i+1)} \circ \beta_{(i+2)} \circ \beta_{(i+3)} \circ \beta^2_{(i+4)} \circ \beta^4_{(i+5)} \circ C(n, s + 6) + \ldots : i_{s+3} > i_{s+2} = i_{s+1} + 1\right)$$

$$E\left(A(s) \circ \beta^2_{(i+1)} \circ \beta^4_{(i+2)} \circ \beta_{(i+3)} \circ \beta_{(i+4)} \circ \beta^2_{(i+5)} \circ \beta^4_{(i+6)} \circ C(n, s + 7) + \ldots : i_{s+4} > i_{s+3} = i_{s+2} + 2\right).$$

**Proof.** We have the result for $n \leq 1$. For $n \leq 0$, only the first family of Hopf algebras exists, and for $k \leq 0$, $H_{\mathfrak{bo}_k} = H_{\text{KQ}_k}$. We note that
there is no conflict between the many classes of generators we have exhibited. By induction, we assume that $H_\ast \mathfrak{bo}_k$ is a subalgebra of $H_\ast KQ_k \otimes H_\ast H_k$, as stated. The bar spectral sequence for $H_\ast \mathfrak{bo}_{k+1}$ collapses, because the bar spectral sequence for $H_\ast (KQ_{k+1} \times H_{k+1})$ is known to collapse; to see this, we only need to verify that the inclusion induces a monomorphism on the $E^2$-terms.

Each listed Hopf algebra in $H_\ast \mathfrak{bo}_k$ gives rise to one or two Hopf algebras in $H_\ast \mathfrak{bo}_{k+1}$. In $H_\ast \mathfrak{bo}_k$ there is exactly one algebra from the first family. It is a subalgebra of $H_\ast KQ_k$, and we see from the steps already given how it gives rise to a first family algebra in $H_\ast \mathfrak{bo}_{k+1}$, with the appropriate restriction on $\alpha(i)$. In addition, in half the cases it spawns a polynomial algebra in the second family, with the appropriate condition on $\alpha(i) + j$.

We treat each algebra of the third or fourth families in $H_\ast \mathfrak{bo}_k$ as a subalgebra of $H_\ast H_k$, and we have already noted how such an algebra gives rise to an algebra of the fourth family in $H_\ast \mathfrak{bo}_{k+1}$.

The only case that requires any discussion is how an algebra from the second family in $H_\ast \mathfrak{bo}_k$ gives rise to an algebra from the third family in $H_\ast \mathfrak{bo}_{k+1}$. Because the generators decompose in $H_\ast KQ_k$, we must map them into $H_\ast H_k$ instead. Thus we need to compute $\Theta\left(F^\bar{j}(\xi_{\bar{i}} \circ [\lambda^{-n}])\right)$, etc.

We need $\Theta\left(F^\bar{m}(\xi_k \circ [\lambda^{-n}])\right)$ whenever $\alpha(k) + m = 4n$ and $m \geq 1$. We write $k = 2^i(2q + 1)$, so that $\alpha(k) = \alpha(q) + 1$. By step 8, we have

$$V^\bar{j}\Theta\left(F^\bar{m}(\xi_{k} \circ [\lambda^{-n}])\right) = \Theta(e \circ \xi_{\bar{i}} \circ [\alpha\lambda^{-n}]),$$

where

$$i = 2^m(2q + 1) - 1 = (2^m - 1) + 2^{m+1}q,$$
so that \(i\) is odd and \(\alpha(i) = \alpha(g) + m = 4n - 1\). We can evaluate this by factoring \(e \circ z_i \circ [\alpha \lambda^{-n}]\) in \(H.\mathfrak{g}_\mathfrak{s}\). Let

\[
i = (2^{i_1} + 2^{i_2+1} + 2^{i_3+2}) + (2^{i_4} + 2^{i_5} + 2^{i_6+1} + 2^{i_7+2}) + \ldots
\]

\[
+ (2^{i_{4n-4}} + 2^{i_{4n-3}} + 2^{i_{4n-2}+1} + 2^{i_{4n-1}+2})
\]

be the binary expansion of \(i\), so that \(i_1 = 0, i_5 > i_4 \geq i_3 + 3, i_9 > i_8 \geq i_7 + 3, \ldots\)

Then

\[
e \circ z_i \circ [\alpha \lambda^{-n}] = e \circ (z_{n_1} \circ [\alpha \lambda^{-1}]) \circ (z_{n_2} \circ [\lambda^{-1}]) \circ \ldots \circ (z_{n_n} \circ [\lambda^{-1}]),
\]

where \(n_1 = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2}, n_2 = 2^{i_4} + 2^{i_5} + 2^{i_6+1} + 2^{i_7+2}, \ldots\). By steps 7 and 8,

\[
\Theta(e \circ z_i \circ [\alpha \lambda^{-n}]) = \beta(0) \circ \left(\beta(0) \circ \beta^2_{(i_2)} \circ \beta^4_{(i_3)}\right) \circ \left(\beta(i_4) \circ \beta(i_5) \circ \beta^2_{(i_6)} \circ \beta^4_{(i_7)}\right) \circ \ldots
\]

\[
= \beta^2_{(0)} \circ \beta^2_{(i_2)} \circ \beta^4_{(i_3)} \circ c(n - 1, 4),
\]

where \(c(n - 1, 4)\) denotes a typical generator of \(C(n - 1, 4)\). Then

\[
\Theta \left(\Gamma^m(z_k \circ [\lambda^{-n}])\right) = \beta^2_{(j)} \circ \beta^2_{(i_2)} \circ \beta^4_{(i_3)} \circ c'(n - 1, 4) + \ldots,
\]

where \(c'(n - 1, 4)\) denotes \(c(n - 1, 4)\) with all indices raised by \(j\).

In the bar spectral sequence for \(H.\mathfrak{g}_\mathfrak{s}^{n+1}\), the polynomial algebra

\[
P \left(\Gamma^m(z_k \circ [\lambda^{-n}]) + \ldots : \alpha(k) + m = 4n\right)
\]

gives the factor

\[
E \left(\sigma(\beta^2_{(i_1)} \circ \beta^2_{(i_2)} \circ \beta^4_{(i_3)} \circ C(n-1, 4))\right)
\]

in the \(E^2\)-term, which corresponds to

\[
E \left(\beta(0) \circ \beta^2_{(i_2)} \circ \beta^2_{(i_3)} \circ \beta^4_{(i_4)} \circ C(n-1, 5)\right)
\]
in $H_*b_{2n+1}$. 

Next, we need $\Theta(F^j(e \circ z_k \circ [\lambda^{-n}]))$ for $\alpha(i) + j = 4n, j \geq 1$. By step 1, $F^j(e \circ z_k) = e \circ z_k$, where $k = 2^j(2i + 1) - 1 = (2^j - 1) + 2^{i+1}i$, so that $\alpha(k) = j + \alpha(i) = 4n$. Again we factor $e \circ z_k \circ [\lambda^{-n}]$ in $H_*b_0$, and find

$$\Theta(F^j(e \circ z_k \circ [\lambda^{-n}])) = \beta(0) \circ \left( \beta(i) \circ \beta(i_2) \circ \beta(i_3) \circ \beta(i_4) \circ \ldots \right) \circ \ldots,$$

where we use the binary expansion

$$k = (2^{i_1} + 2^{i_2} + 2^{i_3+1} + 2^{i_4+2}) + (2^{i_5} + 2^{i_6} + 2^{i_7+1} + 2^{i_8+2}) + \ldots$$

Since $j \geq 1$, $k$ is odd and $i_1 = 0$. We obtain the factor

$$E\left(\beta^{33}(0) \circ \beta(i) \circ \beta^{22}(i_2) \circ \beta^{11}(i_3) \circ \beta^{00}(i_4) \circ C(n - 1, 5) + \ldots : i_2 > 0\right)$$

in $H_*b_{2n+2}$.

The case $\Theta\left(F^j(e^{o2} \circ z_{4i} \circ [\lambda^{-n}])\right)$ is almost identical to the previous case, with indices doubled. We use $F^j(e^{o2} \circ z_{4i}) = e^{o2} \circ z_{2k}$, where $k$ is odd as above. This time, to evaluate $\Theta(e^{o2} \circ z_{2k} \circ [\lambda^{-n}])$, we need the binary expansion

$$2k = (2^{i_1} + 2^{i_2} + 2^{i_3+1} + 2^{i_4+2}) + (2^{i_5} + \ldots) + \ldots,$$

so that $i_1 = 1$ and $i_2 > 1$. We obtain the factor

$$E\left(\beta^{33}(0) \circ \beta(i) \circ \beta^{22}(i_2) \circ \beta^{11}(i_3) \circ \beta^{00}(i_4) \circ C(n - 1, 5) + \ldots : i_2 > i_1 = 1\right)$$

in $H_*b_{2n+3}$.

Finally, we need $\Theta\left(F^j(\tau_{4i} \circ [\lambda^{-(n+1)}])\right)$ whenever $\alpha(i) + j = 4n + 1$ and $i, j \geq 1$. We reindex and note that by step 12

$$V^j\Theta\left(F^m(\tau_{2j(8q+4)} \circ [\lambda^{-(n+1)}]) + \ldots \right) = \Theta\left(F^m(\tau_{8q+4} \circ [\lambda^{-(n+1)}]) + \ldots \right) = \Theta\left(e^{o4} \circ z_{4i} \circ [\lambda^{-n}] + \ldots \right),$$
where $i = 2^m(2q + 1) - 1$, so that $\alpha(i) = m + \alpha(q) = 4n$. We factor $e^{34} \circ z_{4i} \circ [\lambda^{-n}]$ in $H_{\sigma^*}$ and use step 8 to obtain

$$\beta^{34}_{(0)} \circ \left( \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{2} \circ \beta_{(i_4)}^{34} \right) \circ c(n - 1, 5) + \ldots,$$

where we use the binary expansion

$$4i = (2^h + 2^{i_2} + 2^{i_3+1} + 2^{i_4+2}) + (2^{i_5} + \ldots) + \ldots$$

Since $m \geq 1$, $i$ is odd and $i_2 > i_1 = 2$. Then

$$\Theta \left( F^{m}(\zeta_{2^h(8q+4)} \circ [\beta \lambda^{-(n+1)}]) + \ldots \right)$$

$$= \beta^{34}_{(0)} \circ \left( \beta_{(i_1+i_2)} \circ \beta_{(i_2+i_3)}^{2} \circ \beta_{(i_3+i_4)}^{2} \circ \beta_{(i_4+i_5)}^{34} \right) \circ c'(n - 1, 5) + \ldots$$

and $H_{\sigma^*} \sigma_{8n+5}$ contains the factor (after reindexing)

$$E\left( \beta_{(0)} \circ \beta_{(i_2)}^{34} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{2} \circ \beta_{(i_5)}^{32} \circ \beta_{(i_6)}^{34} \circ C(n - 1, 7) : i_4 > i_3 = i_2 + 2 \right).$$

Thus we have proven our hypothesis. \hfill \blacksquare
3 The Computation of the Hopf Rings

\[ H_* bo\langle 4 \rangle_*, H_* bo\langle 2 \rangle_*, \text{ and } H_* bo\langle 1 \rangle_* \]

3.1 The computation of \( H_* bo\langle 4 \rangle_* \)

To calculate \( H_* bo\langle 4 \rangle_* \), we keep in mind that we have the following exact triangle of spectra:

\[ \Phi : \Sigma^3 K(\mathbb{Z}) \to \Sigma^8 bo \to bo\langle 4 \rangle \to \Sigma^4 K(\mathbb{Z}). \]

We map \( \Sigma^4 K(\mathbb{Z}) \to \Sigma^4 H = \Sigma^4 K(\mathbb{Z}/2) \) to simplify calculations.

Thus we have the maps

\[ \zeta : H_* bo_{n+8} \to H_* bo\langle 4 \rangle_n \]

and

\[ \Theta : H_* bo\langle 4 \rangle_n \to H_* H_{n+4}. \]

We note that all of the generators of \( H_* bo\langle 4 \rangle_n \) either map non-trivially to \( H_* H_{n+4} \) or map from \( H_* bo_{n+8} \), but no element does both.

Our conclusion will be that the maps

\[ bo\langle 4 \rangle_n \to bo_n \to KO_n \]

and

\[ \Theta : bo\langle 4 \rangle_n \to H_{n+4} \]

embed \( H_* bo\langle 4 \rangle_n \) as a sub-Hopf algebra of \( H_* (KO_n \times H_{n+4}) \), which we describe.

We use

\[ \Phi : bo_0 \xrightarrow{\sim} bo\langle 4 \rangle_{-8} = bo_{-8}. \]

We also have the fibration

\[ \Phi : bo_1 \to bo\langle 4 \rangle_{-4} \to K(\mathbb{Z}, 0). \]
We keep in mind that \( bo(4)_{-4} = bo_4 \times \mathbb{Z} = KO_{-4} \). This is our starting place.

Also, \( bo(4)_{-k} = bo_{-k} \) for \( k \leq -4 \).

We recall that

\[
H_\ast H_0 = \mathbb{Z}[z_0, z_0^{-1}],
\]

where \( z_0 = [1] \) and \( z_0^{-1} = [-1] \),

\[
H_\ast bo_4 = \mathcal{P}\left( \mathbb{Z}_4 \circ [\beta \lambda^{-1}] : \alpha(i) \geq 1 \right),
\]

and

\[
H_\ast KO_{-4} = \mathcal{P}\left( \mathbb{Z}_4 \circ [\beta] : i > 0 \right) \otimes \mathcal{P}\left( [\beta], [\beta]^{-1} \right) = H_\ast bo(4)_{-4}.
\]

We clearly have

\[
\zeta : \mathbb{Z}_4 \circ [\beta \lambda^{-1}] \rightarrow \mathbb{Z}_4 \circ [\beta]
\]

for \( i > 0 \), and we also have the maps

\[
\Theta([\beta]) = [1] \text{ and } \Theta([\beta]^{-1}) = [-1] = [1].
\]

The general method of computation involves the same type of argument that we used in the proof of \( H_\ast bo_\ast \). Thus, we use the bar spectral sequence, properties of Hopf rings, and the properties of the Verschiebung and Frobenius maps. We must keep track of the map \( \Theta \) to determine the structure of the exterior algebras in \( H_\ast H_\ast \), as we did in \( H_\ast bo_\ast \). We also use the map \( \zeta \) to determine the structure of the sub-Hopf ring of \( H_\ast bo(4)_\ast \) in \( H_\ast bo_\ast \). We leave all proofs to the reader.

We define

\[
A(s) = \text{the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1,
\]

\[
A(0) = [1],
\]
as before, but now we inductively define

\[ C(n,k) = \text{the set of all } \beta_{(i_k)}^{\alpha_4} \circ \beta_{(i_{k+1})} \circ \beta_{(i_{k+2})} \circ \beta_{(i_{k+3})} \circ C(n-1, k+4), \]

where \( i_{k+2} > i_{k+1} \geq i_k + 3, \) starting from \( C(0, k) = [1] \). As in \( H.b_0 \), we also assume implicitly that \( i_m \geq i_n \geq 0 \) whenever \( m > n \).

Thus \( H.b_0 \) is the tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of \( H.b_0 \):

   \[ P\left( \mathbf{z}_i \circ [\lambda^{-n}] + \ldots : \alpha(i) \geq 4n + 3 \right), \text{on } b_0 \langle 4 \rangle_{8n} \]
   \[ P\left( z_0 \circ [\lambda^{-n}], z_0^{-1} \circ [\lambda^{-n}] \right), \text{on } b_0 \langle 4 \rangle_{8n}, \text{ for } n < 0 \]
   \[ P\left( e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 3 \right), \text{on } b_0 \langle 4 \rangle_{8n+1} \]
   \[ P\left( e^{\alpha_2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 3 \right), \text{on } b_0 \langle 4 \rangle_{8n+2} \]
   \[ E\left( e^{\alpha_3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 3 \right), \text{on } b_0 \langle 4 \rangle_{8n+3} \]
   \[ P\left( z_{4i} \circ [\beta\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 4 \right), \text{on } b_0 \langle 4 \rangle_{8n+4} \]
   \[ P\left( z_0 \circ [\beta\lambda^{-n}], z_0^{-1} \circ [\beta\lambda^{-n}] \right), \text{on } b_0 \langle 4 \rangle_{8n-4}, \text{ for } n \leq 0 \]
   \[ E\left( e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n + 4 \right), \text{on } b_0 \langle 4 \rangle_{8n+5} \]
   \[ E\left( \mathbf{z}_{2i} \circ [\alpha^2\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 5 \right), \text{on } b_0 \langle 4 \rangle_{8n+6} \]
   \[ E\left( [\alpha^2\lambda^{-n}] - 1 \right), \text{on } b_0 \langle 4 \rangle_{8n-2}, \text{ for } n < 0 \]
   \[ E\left( \mathbf{z}_i \circ [\alpha\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 6 \right), \text{on } b_0 \langle 4 \rangle_{8n+7} \]
   \[ E\left( [\alpha\lambda^{-n}] - 1 \right), \text{on } b_0 \langle 4 \rangle_{8n-1}, \text{ for } n < 0. \]

2. Polynomial algebras on generators that decompose in \( H.b_0 \), companions
to the polynomial algebras in the first family:

\[
P \left( F^j \left( \mathbb{Z}_i \circ \left[ \lambda^{-n} \right] \right) + \ldots : \alpha(i) + j = 4n + 3, i, j \geq 1 \right), \text{on } bo(4)_{8n}
\]

\[
P \left( F^j \left( e \circ z_{2i} \circ \left[ \lambda^{-n} \right] \right) + \ldots : \alpha(i) + j = 4n + 3, j \geq 1 \right), \text{on } bo(4)_{8n+1}
\]

\[
P \left( F^j \left( e^{o_2} \circ z_{4i} \circ \left[ \lambda^{-n} \right] \right) + \ldots : \alpha(i) + j = 4n + 3, j \geq 1 \right), \text{on } bo(4)_{8n+2}
\]

\[
P \left( F^j \left( \mathbb{Z}_4i \circ \left[ \beta \lambda^{-\left(n+1\right)} \right] \right) + \ldots : \alpha(i) + j = 4n + 4, i, j \geq 1 \right), \text{on } bo(4)_{8n+3}
\]

3. Exterior algebras involving \( \beta(0) \) that arise from the second family:

\[
E \left( \beta(0) \circ \beta_{(i_2)}^{o_2} \circ \beta_{(i_3)}^{o_2} \circ C(n, 4) + \ldots \right), \text{on } bo(4)_{8n+1}
\]

\[
E \left( \beta_{(i_2)}^{o_3} \circ \beta_{(i_3)}^{o_2} \circ C(n, 4) + \ldots : i_2 > 0 \right), \text{on } bo(4)_{8n+2}
\]

\[
E \left( \beta(0) \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{o_2} \circ C(n, 5) + \ldots : i_3 > i_2 = 1 \right), \text{on } bo(4)_{8n+3}
\]

\[
E \left( \beta(0) \circ \beta_{(i_2)}^{o_4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{o_2} \circ C(n, 6) + \ldots : i_4 > i_3 = i_2 + 2 \right), \text{on } bo(4)_{8n+5}
\]

4. General exterior algebras that arise from the third family by unlimited suspension:

\[
E \left( A(s) \circ \beta_{(i_{s+1})}^{o_2} \circ \beta_{(i_{s+2})}^{o_2} \circ \beta_{(i_{s+3})}^{o_2} \circ C(n, s + 4) + \ldots \right)
\]

\[
E \left( A(s) \circ \beta_{(i_{s+1})}^{o_4} \circ \beta_{(i_{s+2})} \circ \beta_{(i_{s+3})}^{o_2} \circ C(n, s + 4) + \ldots : i_{s+2} > i_{s+1} \right)
\]

\[
E \left( A(s) \circ \beta_{(i_{s+1})}^{o_4} \circ \beta_{(i_{s+2})} \circ \beta_{(i_{s+3})} \circ \beta_{(i_{s+4})}^{o_2} \circ C(n, s + 5) + \ldots : i_{s+3} > i_{s+2} = i_{s+1} + 1 \right)
\]

\[
E \left( A(s) \circ \beta_{(i_{s+1})}^{o_2} \circ \beta_{(i_{s+2})}^{o_4} \circ \beta_{(i_{s+3})} \circ \beta_{(i_{s+4})} \circ \beta_{(i_{s+5})}^{o_2} \circ C(n, s + 6) + \ldots : i_{s+4} > i_{s+3} = i_{s+2} + 2 \right).
\]

### 3.2 The computation of \( H_* bo(2)_* \)

To calculate \( H_* bo(2)_* \), we use the following exact triangle of spectra:

\[
\Phi : \Sigma H \rightarrow bo(4) \rightarrow bo(2) \rightarrow \Sigma^2 H.
\]
Once again we define
\[
\Theta : H_* \overline{bo(2)}_n \to H_* \overline{H}_{n+2}
\]
and
\[
\zeta : H_* \overline{bo(4)}_n \to H_* \overline{bo(2)}_n.
\]
The starting point is
\[
\Phi : bo(4)_{-2} \to bo(2)_{-2} \to H_0,
\]
where we note that
\[
H_* \overline{bo(2)}_{-2} = H_* KO_{-2} = E \left( \bar{\varphi}_{2i} \circ [\alpha^2] : i > 0 \right) \otimes E \left( [\alpha^2] - 1 \right).
\]
Also, \( bo(2)_k = \overline{bo}_k \) for \( k \leq -2 \). Thus we start with
\[
\Theta \left( [\alpha^2] - 1 \right) = [1]
\]
and
\[
\zeta : \bar{\varphi}_{2i} \circ [\alpha^2] \mapsto \bar{\varphi}_{2i} \circ [\alpha^2],
\]
for \( i > 0 \).
Again we define
\[
A(s) = \text{the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1,
\]
\[
A(0) = [1],
\]
but now we inductively define
\[
C(n, k) = \text{the set of all } \beta_{(i_k)}^2 \circ \beta_{(i_{k+1})}^4 \circ \beta_{(i_{k+2})} \circ \beta_{(i_{k+3})} \circ C(n-1, k+4),
\]
where \( i_{k+3} > i_{k+2} \geq i_{k+1} + 3 \), starting from \( C(0, k) = [1] \).
Our conclusion is that the maps

\[ bo(2)_n \to bo_n \to KO_n \]

and

\[ \Theta : bo(2)_n \to H_{n+2} \]

embed \( H_* bo(2)_n \) as a sub-Hopf algebra of \( H_*(KO_n \times H_{n+2}) \), which we describe. Thus \( H_* bo(2)_n \) is the tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of \( H_* bo_\ast \):

\[
\begin{align*}
    P\left( z_i \circ [\lambda^{-n}] + \ldots : \alpha(i) \geq 4n + 2 \right), & \text{on } bo(2)_{sn} \\
    P\left( z_0 \circ [\lambda^{-n}] \right), & \text{on } bo(2)_{sn}, \text{ for } n < 0 \\
    P\left( e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 2 \right), & \text{on } bo(2)_{sn+1} \\
    P\left( e^{2i} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 2 \right), & \text{on } bo(2)_{sn+2} \\
    E\left( e^{3i} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 2 \right), & \text{on } bo(2)_{sn+3} \\
    P\left( z_0 \circ [\beta \lambda^{-n}] \right), & \text{on } bo(2)_{sn+4} \\
    P\left( z_0 \circ [\beta \lambda^{-n}] \right), & \text{on } bo(2)_{sn+4}, \text{ for } n \leq 0 \\
    E\left( e \circ z_{4i} \circ [\beta \lambda^{-n}] : \alpha(i) \geq 4n + 3 \right), & \text{on } bo(2)_{sn+5} \\
    E\left( z_{2i} \circ [\alpha^2 \lambda^{-n}] + \ldots : \alpha(i) \geq 4n + 4 \right), & \text{on } bo(2)_{sn+6} \\
    E\left( [\alpha^2 \lambda^{-n}] - 1 \right), & \text{on } bo(2)_{sn-2}, \text{ for } n \leq 0 \\
    E\left( z_{i} \circ [\alpha \lambda^{-n}] + \ldots : \alpha(i) \geq 4n + 5 \right), & \text{on } bo(2)_{sn+7} \\
    E\left( [\alpha \lambda^{-n}] - 1 \right), & \text{on } bo(2)_{sn-1}, \text{ for } n < 0.
\end{align*}
\]

2. Polynomial algebras on generators that decompose in \( H_* bo_\ast \), companions
to the polynomial algebras in the first family:

\[ P \left( F^j (\bar{z}_i \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n + 2, i, j \geq 1 \right), \text{on } bo\langle 2 \rangle_{8n} \]

\[ P \left( F^j (e \circ \bar{z}_2i \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n + 2, j \geq 1 \right), \text{on } bo\langle 2 \rangle_{8n+1} \]

\[ P \left( F^j (e^{2^i} \circ \bar{z}_{4i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n + 2, j \geq 1 \right), \text{on } bo\langle 2 \rangle_{8n+2} \]

\[ P \left( F^j (\bar{z}_{4i} \circ [\beta \lambda^{-(n+1)}]) + \ldots : \alpha(i) + j = 4n + 3, i, j \geq 1 \right), \text{on } bo\langle 2 \rangle_{8n+4}. \]

3. Exterior algebras involving \( \beta(0) \) that arise from the second family:

\[ E \left( \beta(0) \circ \beta_{(t_2)}^{2^3} \circ C(n, 3) + \ldots \right), \text{on } bo\langle 2 \rangle_{8n+1} \]

\[ E \left( \beta(0) \circ \beta_{(t_2)}^{2^3} \circ \beta_{(t_3)} \circ C(n, 3) + \ldots : i_2 > 0 \right), \text{on } bo\langle 2 \rangle_{8n+2} \]

\[ E \left( \beta_{(t_2)}^{2^3} \circ \beta_{(t_3)} \circ C(n, 4) + \ldots : i_3 > i_2 = 1 \right), \text{on } bo\langle 2 \rangle_{8n+3} \]

\[ E \left( \beta(0) \circ \beta_{(t_2)}^{2^4} \circ \beta_{(t_4)} \circ C(n, 5) + \ldots : i_4 > i_3 = i_2 + 2 \right), \text{on } bo\langle 2 \rangle_{8n+5}. \]

4. General exterior algebras that arise from the third family by unlimited suspension:

\[ E \left( A(s) \circ \beta_{(t_3)}^{2^2} \circ \beta_{(t_2)}^{2^2} \circ C(n, s + 3) + \ldots \right) \]

\[ E \left( A(s) \circ \beta_{(t_3)}^{2^4} \circ \beta_{(t_2)} \circ C(n, s + 3 + \ldots) : i_{s+2} > i_{s+1} \right) \]

\[ E \left( A(s) \circ \beta_{(t_3)}^{2^4} \circ \beta_{(t_2)} \circ \beta_{(t_4)} \circ C(n, s + 4 + \ldots : i_{s+3} > i_{s+2} = i_{s+1} + 1 \right) \]

\[ E \left( A(s) \circ \beta_{(t_3)}^{2^4} \circ \beta_{(t_2)} \circ \beta_{(t_4)} \circ \beta_{(t_5)} \circ C(n, s + 5 + \ldots : i_{s+4} > i_{s+3} = i_{s+2} + 2 \right). \]

3.3 The computation of \( H_*bo\langle 1 \rangle_* \)

To calculate \( H_*bo\langle 1 \rangle_* \), we keep in mind that we have the following exact triangle of spectrums:

\[ \Phi : H \rightarrow bo\langle 2 \rangle \rightarrow bo\langle 1 \rangle \rightarrow \Sigma^1 H. \]
We again define
\[ \Theta : H_*bob(1)_n \to H_*H_{n+1} \]
and
\[ \zeta : H_*bob(2)_n \to H_*bob(1)_n. \]
The starting point is
\[ \Phi : bob(2)_{-1} \to bob(1)_{-1} \to H_0, \]
where we note that
\[ H_*bob(1)_{-1} = H_*KO_{-1} = E\left( \beta_i \circ |\alpha| : i > 0 \right) \otimes E\left( |\alpha| - 1 \right). \]
Also, \( bob(1)_k = \text{bob}_k \) for \( k \leq -1 \).
We define
\[ A(s) = \text{the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1, \]
\[ A(0) = [1] \]
as before, but now we inductively define
\[ C(n, k) = \text{the set of all } \beta_{(i_k)} \circ \beta^{o_2}_{(i_{k+1})} \circ \beta^{o_4}_{(i_{k+2})} \circ \beta_{(i_{k+3})} \circ C(n - 1, k + 4), \]
where \( i_{k+3} \geq i_{k+2} + 3 \) and \( i_k > i_{k-1} \), starting from \( C(0, k) = [1] \).
Our conclusion is that the maps
\[ bob(1)_n \to \text{bob}_n \to KO_n \]
and
\[ \Theta : bob(1)_n \to H_{n+1} \]
embed \( H_*bob(1)_n \) as a sub-Hopf algebra of \( H_* (KO_n \times H_{n+1}) \), which we describe. Thus \( H_*bob(1)_n \) is the tensor product of the following four families of Hopf algebras:
1. Polynomial and exterior subalgebras of $H_*b_*$:

\[ P(\bar{x}_i \circ [\lambda^{-n}] + \ldots : \alpha(i) \geq 4n + 1), \text{ on } bo\langle 1 \rangle_{8n} \]
\[ P(\bar{z}_0 \circ [\lambda^{-n}], \bar{z}_0^{-1} \circ [\lambda^{-n}]), \text{ on } bo\langle 1 \rangle_{8n}, \text{ for } n < 0 \]
\[ P(\epsilon \circ \bar{z}_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 1), \text{ on } bo\langle 1 \rangle_{8n+1} \]
\[ P(\epsilon^{o2} \circ \bar{z}_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 1), \text{ on } bo\langle 1 \rangle_{8n+2} \]
\[ E(\epsilon^{o3} \circ \bar{z}_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n + 1), \text{ on } bo\langle 1 \rangle_{8n+3} \]
\[ P(\bar{z}_{4i} \circ [\beta \lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 2), \text{ on } bo\langle 1 \rangle_{8n+4} \]
\[ P(\bar{z}_0 \circ [\beta \lambda^{-n}], \bar{z}_0^{-1} \circ [\beta \lambda^{-n}]), \text{ on } bo\langle 1 \rangle_{8n-4}, \text{ for } n \leq 0 \]
\[ E(\epsilon \circ \bar{z}_{4i} \circ [\beta \lambda^{-(n+1)}] : \alpha(i) \geq 4n + 2), \text{ on } bo\langle 1 \rangle_{8n+5} \]
\[ E(\bar{z}_{2i} \circ [\alpha^2 \lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 3), \text{ on } bo\langle 1 \rangle_{8n+6} \]
\[ E(\bar{z}_i \circ [\alpha^2 \lambda^{-n}] - 1), \text{ on } bo\langle 1 \rangle_{8n-2}, \text{ for } n \leq 0 \]
\[ E(\bar{z}_i \circ [\alpha \lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n + 4), \text{ on } bo\langle 1 \rangle_{8n+7} \]
\[ E(\bar{z}_i \circ [\alpha \lambda^{-n}] - 1), \text{ on } bo\langle 1 \rangle_{8n-1}, \text{ for } n \leq 0. \]

2. Polynomial algebras on generators that decompose in $H_*b_*$, companions to the polynomial algebras in the first family:

\[ P(F^j(\bar{x}_i \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n + 1, i, j \geq 1), \text{ on } bo\langle 1 \rangle_{8n} \]
\[ P(F^j(\epsilon \circ \bar{z}_{2i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n + 1, j \geq 1), \text{ on } bo\langle 1 \rangle_{8n+1} \]
\[ P(F^j(\epsilon^{o2} \circ \bar{z}_{4i} \circ [\lambda^{-n}] + \ldots : \alpha(i) + j = 4n + 1, j \geq 1), \text{ on } bo\langle 1 \rangle_{8n+2} \]
\[ P(F^j(\bar{z}_{4i} \circ [\beta \lambda^{-(n+1)}]) + \ldots : \alpha(i) + j = 4n + 2, i, j \geq 1), \text{ on } bo\langle 1 \rangle_{8n+4}. \]
3. Exterior algebras involving $\beta_{(0)}$ that arise from the second family:

$$E\left(\beta_{(0)}^{\circ 3} \circ C(n, 2) + \ldots \right), \text{ on } bo\langle 1 \rangle_{8n+2}$$

$$E\left(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ C(n, 3) + \ldots : i_2 = 1 \right), \text{ on } bo\langle 1 \rangle_{8n+3}$$

$$E\left(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ C(n, 4) + \ldots : i_3 = i_2 + 2 \right), \text{ on } bo\langle 1 \rangle_{8n+5}$$

$$E\left(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} \circ \beta_{(i_5)} \circ C(n, 6) + \ldots : i_5 \geq i_4 + 3 \right), \text{ on } bo\langle 1 \rangle_{8n+9}.$$  

4. General exterior algebras that arise from the third family by unlimited suspension:

$$E\left(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ C(n, s + 2) + \ldots \right)$$

$$E\left(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})} \circ C(n, s + 3) + \ldots : i_{s+2} = i_{s+1} + 1 \right)$$

$$E\left(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})} \circ \beta_{(i_{s+3})} \circ C(n, s + 4) + \ldots : i_{s+3} = i_{s+2} + 2 \right)$$

$$E\left(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})}^{\circ 2} \circ \beta_{(i_{s+3})}^{\circ 2} \circ \beta_{(i_{s+4})}^{\circ 4} \circ \beta_{(i_{s+5})} \circ C(n, s + 6) + \ldots : i_{s+5} \geq i_{s+4} + 3 \right).$$

3.4 The computation of $H_* bo_*$

We note that we can use this same process to come full circle, calculating $H_* bo_*$ from $H_* bo\langle 1 \rangle_\ast$. Here we start with the exact triangle of spectra

$$\Phi : \Sigma^{-1} K(Z) \to bo\langle 1 \rangle \to bo\langle 0 \rangle \to K(Z),$$

and proceed from this point as we did in the three previous calculations. ■
References


Vita

The author was born in Boston, Massachusetts on Jan. 2, 1971. She attended The State University of New York at Buffalo, receiving a Bachelor of Arts in mathematics. She then attended the Johns Hopkins University, receiving a Ph.D. in mathematics.