The Homology of the Double Loop Space of the Thom Space $MU(n)$

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§0. Introduction

In this paper, we calculate the homology of the double loop space of the Thom space of the classifying space for complex $n$-plane bundles with coefficients $F_2$, $n > 1$, $H_*(\Omega^2 MU(n); F_2)$, where $F_2$ is the field with 2 elements. The main result is as follows.

**Theorem 6.9** $H_*(\Omega^2 MU(n); F_2)$ is a polynomial algebra. The module of generators $QH_*(\Omega^2 MU(n); F_2)$ has a basis isomorphic to $\{[e'_1], [e'_2], \ldots\}$, where $\{e'_1, e'_2, \ldots\}$ is a basis of the primitive module $PCotor^{H_*(MU(n); F_2)}(F_2, F_2)$ and $\deg[e'_i] = \deg[e'_j] = 1$.

We consider the following natural map $f : S^2 MU(n-1) \to MU(n)$. It then induces an injective map $(\Omega f)^* : H^*(\Omega MU(n); F_2) \to H^*(\Omega S^2 MU(n-1); F_2)$ (See Proposition 3.5). Therefore we expect to study $H(\Omega MU(n); F_2)$ and $H(\Omega^2 MU(n); F_2)$ by studying $H(\Omega S^2 MU(n-1); F_2)$ and $H(\Omega^2 S^2 MU(n-1); F_2)$. Using the Eilenberg-Moore spectral sequence, we obtain that $H^*(\Omega S^2 MU(n-1); F_2)$ is an exterior algebra (See Theorem 2.9). Hence $H^*(\Omega MU(n); F_2)$ is an exterior algebra (See Theorem 3.7).

In order to calculate $H^*(\Omega^2 MU(n); F_2)$ by Eilenberg-Moore spectral sequence, we need to obtain generators of $H^*(\Omega MU(n); F_2)$. We notice that since $H_*(\Omega S^2 MU(n-1); F_2)$ is the tensor algebra $T(\sum_{q>0} H_q(SMU(n-1); F_2))$ (see [3]), the primitive module of $H_*(\Omega S^2 MU(n-1); F_2)$ is the free restricted Lie algebra on $H_*(SMU(n-1); F_2)$ (See Proposition 5.6). So we consider the dual of our result and obtain that the map $Cotor^{H_*(S^2 MU(n-1); F_2)}(F_2, F_2) \to Cotor^{H_*(MU(n); F_2)}(F_2, F_2)$ is surjective and that the kernel of the map is the ideal in $T(\sum_{q>0} H_q(SMU(n-1); F_2))$ generated by

\[
\sum_{i_1=1, \ldots, i_{n-1}=1}^{m_1-1, \ldots, m_{n-1}-1} g_{i_1, \ldots, i_{n-1}} \otimes g_{m_1-i_1, \ldots, m_{n-1}-i_{n-1}} : m_j > 1, \text{ for all } j
\]

(See theorem 4.3), where $g_{j_1, \ldots, j_{n-1}} = s^{-1}(b_{j_1} \circ \ldots \circ b_{j_{n-1}})$, $b_{j_1} \circ \ldots \circ b_{j_{n-1}}$ is the basis of $H_*(MU(n-1); F_2)$ and $s$ is the suspension isomorphism $S_p : H_p(SMU(n-1); F_2) \to H_{p-1}(MU(n-1); F_2)$. Denote $H' = Cotor^{H_*(MU(n); F_2)}(F_2, F_2)$. We have the homomorphism $H_*(\Omega S^2 MU(n-1); F_2) \to H'$ induces a surjective homomorphism on their primitive module $PH_*(\Omega S^2 MU(n-1); F_2) \to PH'$ (See Theorem 5.7). Furthermore the primitive
module $PH'$ is spanned by $LG' \cup \{g_1, \ldots, 1\}$ as a vector space, where $LG'$ is the restricted Lie algebra on $G'$ and $G'$ is a vector space spanned by $\{g_{i_1, \ldots, i_{n-1}} : 0 < i_1 \leq \ldots \leq i_{n-1}, i_{n-1} > 1\}$ (See Theorem 5.12).

We denote by $EPH'$ the exterior algebra on $PH'$, which is isomorphic to $H_*(\Omega MU(n); F_2)$ as a coalgebra by the Poincaré-Birkhoff-Witt Theorem. Since the computation of the homology of the Eilenberg-Moore spectral sequence only needs the coalgebra structure, we can show the theorem 6.9.

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§1. The Eilenberg-Moore Spectral Sequence

In the remaining sections, $K$ always denotes the field $F_2$, and we abbreviate $H^*(\ , K)$ to $H^*(\ )$, etc.

Let $A$ be a differential graded augmented algebra over $K$ with differential $\partial$. Define $B^{-k}(A, K) = A \otimes \tilde{A} \otimes \cdots \otimes \tilde{A}$, where $\tilde{A} = \text{Ker}[\varepsilon : A \to K]$. It is customary to denote $B^{-k}(A, K)$ as $B^{-k}$, a generating element of $B^{-k}$ as $a_0[a_1|a_2|\cdots|a_k]$, and an element of $B^0$ as $a_0[\ ]$. Define $\delta : B^{-k} \to B^{-k+1}$, $\partial : B^{-k,n} \to B^{-k,n+1}$ and $\varepsilon : B^0 \to K$, as follows,

$$\delta(a_0[a_1|\cdots|a_k]) = a_0a_1[a_2|\cdots|a_k] + \sum_{i=1}^{k-1} a_0[a_1|\cdots|a_i a_{i+1}|\cdots|a_k],$$

$$\partial(a_0[a_1|\cdots|a_k]) = (\partial a_0)[a_1|\cdots|a_k] + \sum_{i=1}^{k} a_0[a_1|\cdots|\partial a_i|\cdots|a_k]$$

and

$$\varepsilon(a_0[\ ])) = \varepsilon(a_0).$$

We can check that $\delta \delta = 0, \partial \partial = 0, \partial \delta + \delta \partial = 0$ and that $\delta$ is an $A$-morphism.

**Definition 1.1** $\text{Tor}_A(K, K) = H(\overline{B^*}, D)$, where $\overline{B^*} = K \otimes_A B^*$ and the $K$-graded differential module $(B^*, D)$ formed by letting $(B^*)^j = \bigoplus_{n+m=j} (B^m)^n$, $D = \delta + \partial$. 

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Let $\Sigma_{p+q}$ be the group of permutations of the set $\{1,2,\ldots,p+q\}$. Any $\sigma \in \Sigma_{p+q}$ is called a $(p,q)$-shuffle if the following hold:

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p),$$

$$\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q).$$

Suppose

$$[a_1|a_2|\cdots|a_k] \in B^{-k}, \quad [b_1|b_2|\cdots|b_s] \in B^{-s}.$$

Define the shuffle product

$$* : \ B^{-k} \otimes B^{-s} \rightarrow B^{-k-s}$$

by

$$[a_1|a_2|\cdots|a_k] * [b_1|b_2|\cdots|b_s] = \sum_{(k,s)-\text{shuffles } \sigma} [w_{\sigma^{-1}(1)}|w_{\sigma^{-1}(2)}|\cdots|w_{\sigma^{-1}(k+s)}],$$

where $w_i = a_i$ for $i \leq k$, $w_j = b_{j-k}$ for $k < j \leq k+s$. The shuffle product induces a multiplication

$$* : \ Tor_A(K,K) \otimes Tor_A(K,K) \rightarrow Tor_A(K,K).$$

We also can define the coproduct

$$\Delta : \ B^n \rightarrow \bigoplus_{r+s=n} B^r \otimes B^{-s}$$

on a typical element, $[a_1|a_2|\cdots|a_n]$, by

$$\Delta[a_1|a_2|\cdots|a_n] = \sum_{j=0}^n [a_1|a_2|\cdots|a_j] \otimes [a_{j+1}|a_{j+2}|\cdots|a_n].$$

**Theorem 1.2** If $A$ is a graded differential algebra over $K$, then with the $*$ multiplication and $\Delta$ comultiplication, $B^* \otimes$ is a differential Hopf algebra and this induces the structure of Hopf algebra on $Tor_A(K,K)$ with multiplication, $*$, as given above.

**Proof** See [1] p241 (7.15).

**Theorem 1.3** (Eilenberg-Moore) There is a spectral sequence, lying in the second quadrant, with $E_2^{*,*} = Tor_{*H(A)}^{*}(K,K)$, which converges to $Tor_A^{*,*}(K,K)$, where $A$ is a differential graded algebra over $K$. 

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**Proof**  See [1] p226 (7.5).

**Theorem 1.4**  *The Eilenberg-Moore spectral sequence*

\[ \text{Tor}_{H(A)}(K, K) \Rightarrow \text{Tor}_A(K, K) \]

is a spectral sequence of Hopf algebras, converging to its target as a Hopf algebra. The natural Hopf algebra structure on \( \text{Tor}_{H(A)}(K, K) \) agrees with the Hopf algebra structure induced by the spectral sequence.

**Proof**  Since the spectral sequence is induced by the filtration \( F^{-k}(B^*) = \bigoplus_{-s \geq -k} B^{-s} \), it is easy to check that the multiplication and comultiplication are compatible with the filtration. Since \( \frac{E^{-s}}{F^{-s}} = B^{-s} \), we have

\[ E_1^{-s,*} = H^{-s+s}(\frac{F^{-s}}{F^{-s+1}}; D) = H^*(\overline{B}^{-s}; \partial), \]

where \( \partial \) is the differential on the tensor product \( \tilde{A} \otimes \tilde{A} \otimes \cdots \otimes \tilde{A} \), i.e., \( E_1^{-s,*} = \overline{B}^{-s}(\tilde{H}(A)) \) by the Künneth theorem. By the map

\[ \theta : \overline{B}^{-s}(\tilde{H}(A)) \to H(\overline{B}^{-s}(A)) \]

in the Künneth theorem, the product induced by the shuffle product of \( B^* \) in \( E_1 \) coincides with natural shuffle product defined on \( B^*(\tilde{H}(A)) \).

Using the Künneth theorem, we also can prove that the coproduct induced by the coproduct of \( B^* \) in \( E_1 \) coincides with natural coproduct defined on \( B^*(\tilde{H}(A)) \). Hence the natural Hopf algebra structure on \( \text{Tor}_{H(A)}(K, K) \) coincides with the Hopf algebra structure induced by spectral sequence.

**Q.E.D.**

**Theorem 1.5**  *Eilenberg-Moore*  Suppose \( B \) is a connected pointed topological space with \( H_1(B) = 0 \). Then there is a natural isomorphism of algebra \( \theta^* : \text{Tor}_{C^*}(B)(K, K) \to H^*(\Omega B) \), where \( C^*(B) \) is the cochain complex of \( B \).

**Proof**  See [1] p233 (7.10) and [1] p237 (7.13).

**Corollary 1.6**  If the pointed topological space \( B \) is simply connected, then there is a spectral sequence with \( E_2 = \text{Tor}_{H^*}(B)(K, K) \) converging to \( H^*(\Omega B) \) as a Hopf algebra.

**Proof**  1.6 is an immediate conclusion of 1.5 and 1.4.
Q.E.D.

If $A$ and $A'$ are differential graded augmented algebras over $K$, and $h : A \rightarrow A'$ is a homomorphism of differential graded augmented $K$-algebras, then $h$ induces a map

$$\overline{B}(h) : \overline{B}^*(A, K) \longrightarrow \overline{B}^*(A', K).$$

$\overline{B}(h)$ commutes with $D$ and the coproduct $\Delta$. So it induces a homomorphism of Hopf algebras

$$\text{Tor}_h(1, 1) : \text{Tor}_A(K, K) \longrightarrow \text{Tor}_{A'}(K, K).$$

If we consider the duals of all the above definitions and theorems, etc., we can get a similar result for the homology case.

Let $C$ be a differential graded coaugmented coalgebra over $K$ (see [7] p217) with differential $d$ and coproduct $\Delta : C \rightarrow C \otimes C$. Let $M$ be a right $C$-comodule and $N$ be a left $C$-comodule. Define the cotensor product $M[\ ]_C N = \ker[i : M \otimes N \longrightarrow M \otimes C \otimes N]$, where $i = \Delta_M \otimes 1_N + 1_M \otimes \Delta_N$ with $\Delta_M$ and $\Delta_N$ being the structure morphisms of the comodules.

Put $C = C/K$, so that if $C$ is connected,

$$C_n = \begin{cases} 0, & n = 0, \\ C_n, & n > 0. \end{cases}$$

Let $B_{-r}(C, K) = C \otimes C \otimes \cdots \otimes C$, where $B_0(C, K) = C$, and $B_\bullet(C, K) = \sum_{r \geq 0} B_{-r}(C)$.

Denote $\tilde{B}_{-r} = K[\ ]_C B_{-r}(C, K) = C \otimes \cdots \otimes C$, and $\tilde{B}_\bullet = K[\ ]_C B_\bullet$. The canonical isomorphism $\tilde{B}_{-r}(C) \otimes \tilde{B}_{-s}(C) \rightarrow \tilde{B}_{-r-s}(C)$ induces a product by juxtaposition $\mu : \tilde{B}_\bullet(C) \otimes \tilde{B}_\bullet(C) \rightarrow \tilde{B}_\bullet(C)$. Define $\delta' : \tilde{B}_{-r}(C) \rightarrow \tilde{B}_{-r-1}(C)$ by

$$\delta'(a_1 \otimes a_2 \otimes \cdots \otimes a_r) = \sum_{i=1}^{r} a_1 \otimes \cdots \otimes (\Delta a_i) \otimes \cdots \otimes a_r.$$

Let $\tilde{D}$ be the boundary of $\tilde{B}_\bullet(C)$ defined by

$$\tilde{D}(a_1 \otimes a_2 \otimes \cdots \otimes a_r) = \sum_{i=1}^{r} a_1 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_r + \delta'(a_1 \otimes a_2 \otimes \cdots \otimes a_r).$$
Also we denote $[a_1 | a_2 | \cdots | a_r] = a_1 \otimes \cdots \otimes a_r$.

**Definition 1.1'** (See [2]) $\text{Cotor}^C(K, K) = H(\tilde{B}_*(C), \tilde{D})$.

**Proposition 1.7** Let $(B, b)$ be a 1-connected space with base point. There is a natural algebra structure on $\text{Cotor}^{C_*(B)}(K, K)$, which is induced by $\mu$, and a natural isomorphism $\text{Cotor}^{C_*(B)}(K, K) \cong H_* (\Omega B)$.


1.6' If $B$ is 1-connected, then there is a spectral sequence with

$$E^2 = \text{Cotor}^{H_*(B)}(K, K)$$

converging to $\text{Cotor}^{C_*(B)}(K, K)$.

1.4' The spectral sequence in 1.6' is a spectral sequence of Hopf algebras, converging to its target as Hopf algebra.

If $C$ and $C'$ are differential graded coalgebras over $K$ and $h : C \rightarrow C'$ is a homomorphism of such, then $h$ induces a map $\tilde{B}(h) : \tilde{B}_*(C) \rightarrow \tilde{B}_*(C')$ which commutes with $\tilde{D}$ and $\mu$. So it induces a homomorphism of Hopf algebras

$$\text{Cotor}^h(1, 1) : \text{Cotor}^{C_*(K, K)} \rightarrow \text{Cotor}^{C'_*(K, K)}.$$

§2. The computation of $H^*(\Omega S^2 MU(n)); n \geq 1$.

Let $BU(n)$ be the classifying space for complex $n$-plane bundles (the limit of complex Grassmann manifolds $\lim_{m \rightarrow \infty} G_m (C^n)$), $\gamma^n (C^\infty)$ be the canonical complex $n$-plane bundle over $BU(n)$, $E(\gamma^n)$ be the total space of $\gamma^n$ and $MU(n)$ be the Thom complex of $E(\gamma^n)$.
Theorem 2.1. The cohomology $H^*(BU(n); Z)$ is the polynomial ring over $Z$ generated by Chern classes $c_1, c_2, \ldots, c_n$. There are no polynomial relations among these $n$ generators.


Theorem 2.2. One has an exact sequence

$$0 \leftarrow H^*(BU(n - 1); K) \leftarrow H^*(BU(n); K) \leftarrow H^*(MU(n); K) \leftarrow 0$$

where $\beta$ is identified with zero section, $\beta(u) = c_n$ identifying $H^*(MU(n); K)$ with ideal generated by $c_n$ in $H^*(BU(n); K)$ where $u$ is the Thom class.


Proposition 2.3. Let $X$ be a connected space and $\Lambda$ be a commutative ring (with unit element) such that $H_q(X; \Lambda)$ is a free $\Lambda$-module for each $q \geq 0$. Then we have a natural isomorphism of $H_*(\Omega S^2(X); \Lambda)$ with the tensor algebra $T(\sum_{q \geq 0} H_q(X; \Lambda))$.

Proof. See [3], p22-07 Corollary 2.

Lemma 2.4

$$H^*(\Omega S^2 MU(n)) \cong \overline{B}(H(S^2 MU(n)))$$

as vector space.

Proof. By Proposition 2.3, we have

$$H_*(\Omega S^2 MU(n)) = T(\sum_{q > 0} H_q(SMU(n))).$$

Then

$$H^*(\Omega S^2 MU(n)) = (T(\sum_{q > 0} H_q(SMU(n))))^*$$

$$= \overline{B}(H^*(S^2 MU(n)))$$

as a vector space.

Q.E.D.

We can use the Eilenberg-Moore spectral sequence to compute $H^*(\Omega S^2 MU(n))$. Since the multiplication on $H^*(S^2 MU(n))$ is trivial, the differential on $\overline{B}(H^*(S^2 MU(n)))$ is trivial. Thus

$$\text{Tor}_{H^*(S^2 MU(n))}(K, K) = \overline{B}(H^*(S^2 MU(n))).$$
By Lemma 2.4, the Eilenberg-Moore spectral sequence

$$\text{Tor}_{H^*}(S^2MU(n))(K, K) \implies \text{Tor}_{C^*}(S^2MU(n))(K, K)$$

collapses. Hence

$$\overline{B}^*(H^*(S^2MU(n))) = H^*(\Omega S^2MU(n)).$$

**Lemma 2.5** Let $A$ be any algebra. We have $\alpha^2 = 0$ for any $\alpha \in \text{Tor}^n_A(K, K)$, where $n > 0$.

**Proof** For any

$$\alpha = [a_1|a_2|\cdots|a_n] \in \overline{B}^*(A),$$

we have

$$\alpha^2 = \sum_{(n,n)-\text{shuffles } \sigma} [w_{\sigma^{-1}(1)}|w_{\sigma^{-1}(2)}|\cdots|w_{\sigma^{-1}(2n)}].$$

For any $(n,n)$-shuffle permutation $\sigma$, there exists one and only one $(n,n)$-shuffle permutation $\sigma'$ such that

$$\begin{align*}
\sigma'(n+i) &= \sigma(i) & i &= 1, 2, \ldots, n; \\
\sigma'(j) &= \sigma(n+j) & j &= 1, 2, \ldots, n.
\end{align*}$$

Then

$$\begin{align*}
[w_{\sigma^{-1}(1)}|w_{\sigma^{-1}(2)}|\cdots|w_{\sigma^{-1}(2n)}] \\
&= [w_{\sigma'^{-1}(1)}|w_{\sigma'^{-1}(2)}|\cdots|w_{\sigma'^{-1}(2n)}].
\end{align*}$$

Since $\sigma \neq \sigma'$, and $K = F_2$, the terms of $\alpha^2$ cancel out in pairs. Then $\alpha^2 = 0$. Thus for any $\alpha \in \overline{B}^*(A)$, we have $\alpha^2 = 0$.

**Q.E.D.**

**Theorem 2.6** (Borel) If $A$ is a connected Hopf algebra over the perfect field $K$, the multiplication in $A$ is commutative, and the underlying graded vector space of $A$ is of finite type, then as an algebra, $A$ is isomorphic with a tensor product $\bigotimes_{i \in I} A_i$ of Hopf algebras $A_i$, where $A_i$ is a Hopf algebra with a single generator $x_i$.

**Proof** See [7] p255 (7.11).

**Lemma 2.7** $\text{Tor}_{H^*}(S^2MU(n))(K, K)$ is an exterior algebra,
Proof  By Theorem 2.6,

\[ \text{Tor}_{H^*}(S^2MU(n)) (K, K) = \bigotimes_{i \in I} A_i. \]

By Lemma 2.5, we have \( x_i^2 = 0 \) where \( x_i \) is the generator of \( A_i \) for \( i \in I \).

Q.E.D.

Lemma 2.8  We have \( \alpha^2 = 0 \), for all \( \alpha \in H^*(\Omega S^2MU(n)) \) with \( \text{deg} \alpha > 0 \).

Proof  Let \( H = H^*(\Omega S^2MU(n)) \), a connected filtered Hopf algebra whose associated graded Hopf algebra \( \text{Gr} H \) is \( E_{\infty} \). The spectral sequence collapses by Lemma 2.4,

\[ \text{Gr} H = E_2 = B^* (H^*(S^2MU(n))). \]

But the primitives in \( B^* (H^*(S^2MU(n))) \) are exactly \( B^{-1} (H^*(S^2MU(n))) \), which is entirely in odd degrees. Thus all nonzero primitives in \( \text{Gr} H \) have odd degree.

Let \( \alpha \) be any primitive in \( H \), with filtration exactly \( -k \). Then \( \alpha \) gives a nonzero element \( \overline{\alpha} \in \text{Gr} H \) which is also primitive. Then

\[ \text{deg} \alpha = \text{deg} \overline{\alpha} \]

must be odd. Hence we have that all nonzero primitives in \( H \) have odd degree.

We use induction on degree to prove the result.

Suppose \( \alpha^2 = 0 \) for any \( \alpha \in H^s \) and \( 0 < s < n \), which is vacuously true for \( n = 1 \). For \( \alpha \in H^n \), we have

\[ \Delta \alpha^2 = 1 \otimes \alpha^2 + \alpha^2 \otimes 1 + \sum_i \alpha_i^2 \otimes \alpha_i'^2, \]

where

\[ \Delta \alpha = \alpha \otimes 1 + 1 \otimes \alpha + \sum \alpha_i' \otimes \alpha_i'' \]

and \( \alpha_i' \in H^s, \quad \alpha_i'' \in H^r \) with \( s + r = n, \quad s > 0, \quad r > 0 \). Then by the inductive hypothesis

\[ \Delta \alpha^2 = 1 \otimes \alpha^2 + \alpha^2 \otimes 1. \]

So \( \alpha^2 \) is primitive, has even degree, and must therefore be zero.

Q.E.D.
**Theorem 2.9** \( H^*(\Omega S^2 MU(n)) \) is an exterior algebra.

**Proof** Since \( H^*(\Omega S^2 MU(n)) \) is a Hopf algebra and \( \alpha^2 = 0 \) for any \( \alpha \in H^*(\Omega S^2 MU(n)) \),

Theorem 2.9 follows from Theorem 2.6 immediately. \( \text{Q.E.D.} \)

§3. The cohomology of the loop space over \( MU(n), n \geq 2 \).

Let \( C_s \ (\subset \overline{B}^*(H^*(MU(n)))) \) be the vector space spanned by

\[
\{ [c_n^{k_n} \cdots c_2^{k_2} c_1^{k_1} | \cdots | c_m^{k_m} \cdots c_2^{k_2} c_1^{k_1}] : k_1 + \ldots + k_m = s \},
\]

\( C_0 = K \)

and \( \hat{C}_1 \) be the vector space spanned by

\[
\{ [c_n^{k_n-1} \cdots c_2^{k_2} c_1^{k_1}] : k_1 + \ldots + k^{n-1} > 0 \}.
\]

\( (C_s, d_1) \) is a subcomplex of \( (\overline{B}^*(H^*(MU(n))), d_1) \) and

\[
\overline{B}^*(H^*(MU(n))) = \bigoplus_{s=0}^{\infty} C_s,
\]

where \( d_1 = \delta^* \) in §1. Here it is convenient to write \( H_s(\ ) \) for the homology of the bar construction.

**Theorem 3.1** For \( s \geq 1 \),

\[
H_s(C_s) = C_1 \otimes \hat{C}_1 \otimes \ldots \otimes \hat{C}_1,
\]

and \( H_s(C_s) \) is a subgroup of \( \overline{B}^s \cap C_s \). \( H_t(C_s) = 0 \) for \( t \neq s \).

**Proof** For \( s = 1 \), we have \( H_1(C_1) = C_1 \). The result holds. Suppose that for \( s - 1 \) the result holds. For \( s \), let \( C'_s \) be the subcomplex of \( C_s \) spanned by

\[
\{ [c_n^{k_n} \cdots c_1^{k_1} | \cdots | c_m^{k_m} \cdots c_1^{k_1}] : k_1^n + \ldots + k_m^n = s, \ \text{and} \ k_m^n > 1 \}
\]
and $h$ be the chain map

$$h : \ C_{s-1} \rightarrow C'_s$$

given by

$$h[c_{n_1}^{k_1} \ldots c_{1_1}^{k_1} | \ldots | c_{m_1}^{k_1} \ldots c_{1_1}^{k_1}]$$

$$= [c_{n_1}^{k_1} \ldots c_{1_1}^{k_1} | \ldots | c_{m_1}^{-1 \ldots 1} | c_{m_1}^{k_1} \ldots c_{1_1}^{k_1}] .$$

$h$ is an isomorphism on the chain level. The short exact sequence of complexes

$$0 \rightarrow C'_s \overset{\alpha}{\rightarrow} C_s \overset{\beta}{\rightarrow} C'_s \rightarrow 0$$

induces a long exact sequence

$$\cdots \rightarrow H_{t+1}(C'_s/C_s) \rightarrow H_t(C'_s) \rightarrow H_t(C_s) \rightarrow H_t(C'_s/C_s) \rightarrow H_{t-1}(C'_s) \rightarrow \cdots .$$

The theorem follows from the following two lemmas.

**Lemma 3.2**

$$H_s(C'_s/C_s) \cong H_{s-1}(C_{s-1}) \otimes C_1. $$

$$H_t(C'_s/C_s) = 0 \quad \text{for} \quad t \neq s.$$ 

**Proof** Define the chain map

$$g : \ C_{s-1} \otimes C_1 \rightarrow \frac{C_s}{C'_s}$$

by

$$g[c_{n_1}^{k_1} \ldots c_{1_1}^{k_1} | c_{n_2}^{k_2} \ldots c_{1_2}^{k_2} | \ldots | c_{n_m}^{k_1} \ldots c_{1_m}^{k_1}] \otimes [c_{n_1}^{k_1} \ldots c_{1_1}^{k_1} | c_{n_2}^{k_1} \ldots c_{1_2}^{k_1} | \ldots | c_{n_m}^{k_1} \ldots c_{1_m}^{k_1}]$$

$$= [c_{n_1}^{k_1} \ldots c_{1_1}^{k_1} | c_{n_2}^{k_1} \ldots c_{1_2}^{k_1} | \ldots | c_{n_m}^{k_1} \ldots c_{1_m}^{k_1} | c_{n_1}^{k_1} \ldots c_{1_1}^{k_1}].$$

Since $g$ is an isomorphism on the chain level,

$$H(C'_s/C_s) \cong H(C_{s-1} \otimes C_1).$$

By the Künneth theorem, we have
\[ H_t(C_{s-1} \otimes C_1) = H_{t-1}(C_{s-1}) \otimes H_1(C_1) = H_{t-1}(C_{s-1}) \otimes C_1. \]

Thus the lemma holds.

**Lemma 3.3** The map 
\[ H_s\left( \frac{C_s}{C'_s} \right) \xrightarrow{\partial_s} H_{s-1}(C'_s) \]

is surjective and takes \( z \otimes [c_n] \) to \( z \), where
\[ H_{s-1}(C_{s-1}) \otimes C_1 \cong H_s\left( \frac{C_s}{C'_s} \right) \]

and
\[ H_{s-1}(C'_s) \cong H_{s-1}(C_{s-1}). \]

**Proof** Recall that
\[ \partial_s \{b\} = \{\alpha_s^{-1}d_1\beta_s^{-1}b\}. \]

There is an obvious lifting
\[ \tilde{g} : C_{s-1} \otimes C_1 \longrightarrow C_s \]

of \( g \), by
\[ \tilde{g}(x_1 \cdots x_{s-1} \otimes y) = [x_1] \cdots [x_{s-1}]y, \]

which is not a chain map. Instead, from the definition of \( d_1 \),
\[ d_1\tilde{g}(a \otimes [c_n]) = \tilde{g}(d_1a \otimes [c_n]) + h(a), \]

for
\[ a = [c_n]c_{n-1}^{k_{n-1}} \cdots c_1^{k_1} | c_n c_{n-1}^{k_{n-1}} \cdots c_1^{k_1} | \cdots | c_n c_{n-1}^{k_{n-1}} \cdots c_1^{k_1} \]

and therefore for any \( a \in C_{s-1} \) in filtration index \( s-1 \). If \( a \) is a cycle representing \( z \), this gives \( \partial_s g_s(z \otimes [c_n]) = z \).

**Proof of Remainder of Theorem 3.1**
From the long exact sequence and Lemma 3.3,

\[ H_s(C_s) \cong Ker[\partial_s : H_s(C_s/C_s') \longrightarrow H_{s-1}(C_s')] \]
\[ \cong Ker[\partial_s : H_{s-1}(C_{s-1}) \otimes C_1 \longrightarrow H_{s-1}(C_{s-1})] \]
\[ \cong H_{s-1}(C_{s-1}) \otimes \hat{C}_1 \]
\[ \cong C_1 \otimes \hat{C}_1 \otimes \cdots \otimes \hat{C}_1 \otimes \hat{C}_1. \]

Q.E.D.

**Corollary 3.4** Every element of \( Tor_{H^*}(MU(n))(K,K) \) is represented by a cycle in \( \sum_{s=0}^{\infty} B^s \cap C_s \).

**Proof** Since

\[ Tor_{H^*}(MU(n))(K,K) = H(B^\bullet)(H^*(MU(n)))) \]
\[ = H(\sum_{s=0}^{\infty} C_s) \]
\[ = \sum_{s=0}^{\infty} H(C_s), \]

the result follows from Theorem 3.1 at once.

Q.E.D.

Let \( CP^\infty \) be the complex projective space. Since \( BU(1) = CP^\infty \), \( MU(1) \cong BU(1) \) and \( S^2 \subset CP^\infty \), let

\[ f : S^2 MU(n-1) \longrightarrow MU(2) \]

be the map \( g \circ I \), where

\[ I : S^2 MU(n-1) = S^2 \wedge MU(n-1) \longrightarrow CP^\infty \wedge MU(n-1) \]

is the inclusion and

\[ g : CP^\infty \wedge MU(n-1) \longrightarrow MU(n) \]

is induced by the Whitney sum. \( f \) induces a cohomology homomorphism

\[ f^* : H^*(MU(n)) \rightarrow H^*(S^2 MU(n-1)) \]

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with
\[ f^*(c_n^k c_{n-1}^{k-1} \ldots c_1^{k_1}) = \begin{cases} 0, & k^n > 1, \\ i \otimes c_{n-1}^{k_{n-1}+1} \ldots c_1^{k_1}, & k^n = 1, \end{cases} \]

where \( i \) is the generator of \( H^2(S^2) \).

**Proposition 3.5** The map

\[ f^* : H^*(MU(n)) \longrightarrow H^*(S^2MU(n-1)) \]

induces an injective map

\[ \text{Tor}_{H^*(MU(n))}(K, K) \longrightarrow \text{Tor}_{H^*(S^2MU(n-1))}(K, K). \]

**Proof** Since \( \sum_{s=0}^{\infty} B^{-s} \cap C_s \) is spanned by
\[ \{ [c_n c_{n-1}^{k_{n-1}} \ldots c_1^{k_1}] [c_n c_{n-1}^{k_{n-1}} \ldots c_1^{k_1}] \ldots [c_n c_{n-1}^{k_{n-1}} \ldots c_1^{k_1}] : k_i \geq 0 \}, \]

the map
\[ B(f^*) : \sum_{s=1}^{\infty} B^{-s} \cap C_s \longrightarrow B^*(H^*(S^2MU(n-1))) \]

is injective. Also (see §2)
\[ \text{Tor}_{H^*(S^2MU(n-1))}(K, K) = B^* H^*(S^2MU(n-1)). \]

Hence by 3.4
\[ \text{Tor}_{H^*(MU(n))}(K, K) \longrightarrow \text{Tor}_{H^*(S^2MU(n-1))}(K, K) \]

is injective. \[ \text{Q.E.D.} \]

**Theorem 3.6** The spectral sequence
\[ \text{Tor}_{H^*(MU(n))}(K, K) \Longrightarrow \text{Tor}_{C^*(MU(n))}(K, K) \]

collapses.
\textbf{Proof}  We denote the spectral sequence
\[ \text{Tor}_{H^*}(S^2MU(n-1))(K, K) \implies \text{Tor}_{C^*}(S^2MU(n-1))(K, K) \]
as \((E', d')\) with
\[ E'_2 = \text{Tor}_{H^*}(S^2MU(n-1))(K, K) \]
and
\[ \text{Tor}_{H^*}(MU(n))(K, K) \implies \text{Tor}_{C^*}(MU(n))(K, K) \]
as \((E, d)\) with
\[ E_2 = \text{Tor}_{H^*}(MU(n))(K, K). \]
Since the spectral sequence \((E', d')\) collapses by Theorem 2.6, \(d'_k = 0\) for \(k = 1, 2, \ldots\).
Since
\[ f^* : E_2 \longrightarrow E'_2 \]
is injective, \(d_2 = 0\). Then \(E_3 = E_2\) and
\[ f^* : E_3 \longrightarrow E'_3 \]
is injective and \(d_3 = 0\). Inductively, we obtain
\[ d_k = 0 \quad \text{for} \quad k > 2. \]

Thus the spectral sequence \((E, d)\) collapses.
\[ \text{Q.E.D.} \]

\textbf{Theorem 3.7} \(H^*(\Omega MU(n))\) is an exterior algebra.
\textbf{Proof}  Since
\[ (\Omega f)^* : H^*(\Omega MU(n)) \longrightarrow H^*(\Omega S^2MU(n - 1)) \]
is an injective map of Hopf algebras, and \(H^*(\Omega S^2MU(n - 1))\) is an exterior algebra, \(H^*(\Omega MU(n))\) is exterior algebra, by Theorem 2.6.
\[ \text{Q.E.D.} \]

We would like to find a set of exterior generators. To do this we have to dualize.

\[ \S 4. \text{ The homology of the loop space of } MU(n), n \geq 2. \]
Let $b_i \in H_*(CP^\infty)$ be the dual of $c_i^1 \in H^*(CP^\infty)$. It is known that the Whitney sum
\[ CP^\infty \wedge CP^\infty \to MU(2) \]
induces a surjective homomorphism on homology
\[ H_*(CP^\infty \wedge CP^\infty) \to H_*(MU(2)) \]
by
\[ b_i \otimes b_j \mapsto \begin{cases} b_i \circ b_j & \text{if } i \leq j, \\ b_j \circ b_i & \text{if } i > j, \end{cases} \]
and that $\widetilde{H}_*(MU(2))$ has a basis
\[ \{b_i \circ b_j : 0 < i \leq j\}. \]
The notation $\circ$ is from [9]. Inductively the Whitney sum
\[ CP^\infty \wedge MU(n-1) \to MU(n) \]
gives a surjective homomorphism on homology
\[ H_*(CP^\infty \wedge MU(n-1)) \to H_*(MU(n)) \]
by
\[ b_i \otimes b_{i_1} \circ b_{i_2} \circ \ldots b_{i_{n-1}} \mapsto b_{i_1} \circ \ldots b_{i_j} \circ b_i \circ b_{i_{j+1}} \circ \ldots b_{i_{n-1}}, \]
where $i_1 \leq \ldots i_j \leq i \leq i_{j+1} \leq \ldots \leq i_{n-1}$. The basis of $\widetilde{H}_*(MU(n))$ is
\[ \{b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_n} : 0 < b_{i_1} \leq b_{i_2} \leq \ldots \leq b_{i_n}\} \]
The inclusion map
\[ I : S^2 MU(n-1) \to CP^\infty \wedge MU(n-1) \]
induces an injective homomorphism on homology
\[ I_* : H_*(S^2 MU(n-1)) \to H_*(CP^\infty \wedge MU(n-1)) \]
by
\[ I_*(i \otimes b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_{n-1}}) = b_1 \otimes b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_{n-1}}. \]
The map
\[ f: \quad S^2MU(n-1) \to MU(n) \]
induces a homology homomorphism
\[ f_*: \quad H_*(S^2MU(n-1)) \to H_*(MU(n)) \]
with
\[ f_*(i \otimes b_{i_1} \circ \ldots \circ b_{i_{n-1}}) = b_1 \circ b_{i_1} \circ \ldots \circ b_{i_{n-1}}, \]
where \( i \) is the generator of \( H_2(S^2) \).

Put \( G = \tilde{H}_*(SMU(n-1)) \), where \( S \) is the suspension isomorphism
\[ S_p: \quad H_p(SMU(n-1)) \to H_{p-1}(MU(n-1)) \]
for all \( p > 1 \). Denote
\[ S^{-1}(b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_{n-1}}) = g_{i_1,i_2,\ldots,i_{n-1}}. \]
\( G \) has a basis
\[ \{ g_{i_1,i_2,\ldots,i_{n-1}} : 0 < i_1 \leq i_1 \leq \ldots \leq i_{n-1} \}, \]
and
\[ g_{i_1,\ldots,i_j,\ldots,i_{n-1}} = g_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_{n-1}}. \]
By the definition of \( \tilde{B}^* \), we have
\[ \tilde{B}^*(H_*(S^2MU(n-1))) = TG, \]
the tensor algebra on \( G \).

Let \( J \) be the ideal in \( TG \) generated by
\[ \{ \sum_{i_1=1}^{m_1-1} \sum_{i_2=1}^{m_2-1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}-1} g_{i_1,i_2,\ldots,i_{n-1}} \otimes g_{m_1-i_1,m_2-i_2,\ldots,m_{n-1}-i_{n-1}} : m_j > 1, j = 1, 2, \ldots, n-1. \}. \]

**Theorem 4.1** \( \cotensor^{f_\ast}(1, 1) \) is surjective and the kernel of \( \cotensor^{f_\ast}(1, 1) \) contains \( J \).
**Proof** Since

\[ f : S^2MU(n - 1) \rightarrow MU(n) \]

induces an injective homomorphism

\[ \text{Tor} f^* (1, 1) : \text{Tor}_{H^* (MU(n))} (K, K) \rightarrow \text{Tor}_{H^* (S^2MU(n-1))} (K, K), \]

\( f \) induces a surjective homomorphism

\[ \text{Cotor} f^* (1, 1) : \text{Cotor}_{H^* (S^2MU(n-1))} (K, K) \rightarrow \text{Cotor}_{H^* (MU(n))} (K, K). \]

Since

\[ \text{Cotor}_{H^* (S^2MU(n-1))} (K, K) = \tilde{B}^* (H^* (S^2MU(n - 1))), \]

\[ \text{Cotor}_{H^* (MU(n))} (K, K) \cong \frac{\tilde{B}^* (H^* (S^2MU(n - 1)))}{\text{Ker Cotor} f^* (1, 1)}. \]

Since

\[ \tilde{B} (f^*) : \tilde{B}^* (H^* (S^2MU(n - 1))) \rightarrow \tilde{B}^* (H^* (MU(n))) \]

is injective, it induces an isomorphism,

\[ \text{Ker Cotor} f^* (1, 1) \cong \text{Im} \delta' \cap \text{Im} \tilde{B} (f^*), \]

where \( \delta' \) is the differential of the spectral sequence defined on §1.

\[ \text{Cotor}_{H^* (MU(n))} (K, K) \rightarrow H^* (\Omega MU(n)). \]

In \( H^* (CP^\infty) \), we have

\[ \Delta b_n = b_n \otimes 1 + b_{n-1} \otimes b_1 + b_{n-2} \otimes b_2 + \ldots + 1 \otimes b_n. \]

The homomorphism \( I^* \) and Whitney sum are homomorphisms of coalgebras, so is \( f^* \). In \( H^* (MU(n)) \), we therefore have

\[ \Delta (b_2 \circ b_{m_1} \circ \ldots \circ b_{m_{n-1}}) \]

\[ = 1 \otimes (b_2 \circ b_{m_1} \circ \ldots \circ b_{m_{n-1}}) + (b_2 \circ b_{m_1} \circ \ldots \circ b_{m_{n-1}}) \otimes 1 \]

\[ + \sum_{i_1=1, i_2=1, \ldots, i_{n-1}=1} (b_1 \circ b_{i_1} \circ \ldots \circ b_{i_{n-1}}) \otimes (b_1 \circ b_{m_1-i_1} \circ \ldots \circ b_{m_{n-1}-i_{n-1}}) \]

for \( n \geq 2 \). Since

\[ f^*(g_{i_1,i_2,\ldots,i_{n-1}}) = f^* [i \otimes b_{i_1} \circ \ldots \circ b_{i_{n-1}}] = [b_1 \circ b_{i_1} \circ \ldots \circ b_{i_{n-1}}], \]

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we have
\[ \tilde{B}(f_\ast)(J) \subset \text{Im} \delta', \]
as required. Thus the theorem holds.

Q.E.D.

**Lemma 4.2** \( TG/J \) is spanned by
\[ \{ g_{i_1^1, i_2^1, \ldots, i_{n-1}^1} \otimes g_{i_1^2, i_2^2, \ldots, i_{n-1}^2} \otimes \cdots \otimes g_{i_1^r, i_2^r, \ldots, i_{n-1}^r} \}, \]
where \( i_{n-1} > 1 \), for \( r > 1 \); \( 0 < i_1^r \leq \ldots \leq i_{n-1}^r \) all \( r > 0 \)

**Proof** Recall that
\[ g_{1,1,\ldots,1} \otimes g_{1,1,\ldots,1} = 0. \]
For any other \( (m_1, m_2, \ldots, m_{n-1}) \) with \( m_{n-1} > 1 \) and \( 0 < m_1 \leq m_2 \leq \ldots \leq m_{n-1} \), we have
\[
\begin{align*}
g_{m_1, m_2, \ldots, m_{n-1}} & \otimes g_{1,1,\ldots,1} \\
= & g_{1,1,\ldots,1} \otimes g_{m_1, m_2, \ldots, m_{n-1}} \\
+ & \left( \sum_{i_1=1}^{m_1-1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}-1} g_{i_1, i_2, \ldots, i_{n-1}} \otimes g_{m_1-i_1, \ldots, m_{n-1}-i_{n-1}} \\
- & g_{1,\ldots,1} \otimes g_{m_1, \ldots, m_{n-1}} - g_{m_1, \ldots, m_{n-1}} \otimes g_{1,\ldots,1} \right)
\end{align*}
\]
in \( TG/J \). So we can move all \( g_{1,1,\ldots,1} \)'s to the left.

Q.E.D.

**Theorem 4.3** The elements in 4.2 give a basis of \( TG/J \) and \( \text{Cotor}^H_\ast(MU(n))(K,K) \).

**Proof** Comparing Lemma 4.2 and Theorem 3.3, we have that \( TG/J \) and \( \text{Tor}^{H_\ast}(MU(n))(K,K) \) have the same size. Thus \( \text{Ker} \text{Cotor}^{H_\ast}_\ast(1,1) = J \). That means
\[
\frac{\text{TG}}{J} = \text{Cotor}^{H_\ast}(MU(n))(K,K).
\]

Q.E.D.

§5. The primitives of \( H_\ast(\Omega MU(n)), n \geq 2 \).
**Definition 5.1** If \( c \in C \), where \( C \) is an augmented coalgebra over \( K \) and
\[
\Delta c = 1 \otimes c + c \otimes 1,
\]
then \( c \) is called a *primitive* element of \( C \). The set
\[
PC = \{ c : \text{ \( c \) is primitive in \( C \)} \}
\]
is called the *primitive module* over \( K \) of \( C \).

Recall that \( K = F_2 \).

**Definition 5.2** A *restricted Lie algebra* over \( K \) is a Lie algebra \( L \) together with a function
\( \xi : L_n \rightarrow L_{2n} \) satisfying
\[
\xi(x + y) = \xi x + \xi y + [x, y] \]
and
\[
[x, \xi y] = [[x, y], y].
\]

Any algebra \( A \) over \( F_2 \) can be made into a restricted Lie algebra by setting
\[
[x, y] = xy - yx, \quad \xi x = x^2.
\]
The axioms hold since
\[
(x + y)^2 = x^2 + y^2 + [x, y]
\]
and
\[
[x, y^2] = [[x, y], y].
\]

**Proposition 5.3** \( PC \) is a restricted Lie algebra over \( K \).

**Proof** We can check the definition directly.

Q.E.D.

**Definition 5.4** If \( L \) is a restricted Lie algebra over \( K \), the *universal enveloping algebra* of \( L \) is an algebra \( V(L) \) together with a morphism of restricted Lie algebras \( i_L : L \rightarrow V(L) \)
such that if \( A \) is an algebra and \( f : L \to A \) is a morphism of restricted Lie algebras, there is a unique morphism of algebras \( \tilde{f} : V(L) \to A \) such that the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{i_L} & V(L) \\
\downarrow & & \downarrow f \\
A
\end{array}
\]

is commutative.

The universal enveloping algebra is easily constructed. Put

\[
V(L) = T(L)/I
\]

where \( T(L) \) is the tensor algebra of \( L \) and \( I \) is the ideal of \( T(L) \) generated by all the elements

\[
\{x \otimes y + y \otimes x + [x, y] \text{ and } x \otimes x + \xi x\}.
\]

**Proposition 5.5** 
If \( L \) is a restricted Lie algebra over \( K \) with basis

\[
\{x_1, x_2, x_3, \ldots \},
\]

then \( V(L) \) has the basis

\[
\{x_{i_1}x_{i_2} \cdots x_{i_k} : i_1 < i_2 < \cdots < i_k, \ k \geq 0\}
\]


**Proposition 5.6**

\[
P(H_* (\Omega S^2 MU(n - 1))) = L(G),
\]

where \( L(G) \) is the free restricted Lie algebra on \( G \).

**Proof** Since \( G \) generates

\[
H_* (\Omega S^2 MU(n - 1)) = TG,
\]

and \( G \subset PTG \) by definition, \( TG \) is primitively generated. By [7] (6.10), we have

\[
TG = V(PTG).
\]

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By definition, 
\[ V(L(G)) = TG, \]
since both sides have the same universal property. We deduce from 5.5 that 
\[ PTG = L(G). \]

Q.E.D.

Denote 
\[ H' = \text{Cotor}^{H,(MU(n))}(K, K), \]
and 
\[ \overline{f}_* = \text{Cotor}^{f_*}(1, 1) : TG \rightarrow H'. \]
Then 
\[ H' = \frac{TG}{J} \]
is also primitively generated, and So \( H' = VPH' \).

**Theorem 5.7** *The homomorphism*

\[ \overline{f}_* : TG \rightarrow H' \]

*induces a surjective homomorphism*

\[ P(\overline{f}_*) : PT(G) \rightarrow PH'. \]

**Proof** Put \( M = \text{Im}(P\overline{f}_*) \). Since \( PT(G) \) and \( PH' \) are restricted Lie algebras and \( \overline{f}_* \) is a homomorphism of restricted Lie algebras, \( M \) is a restricted Lie algebra. Since 
\( TG = VPTG, \quad H' = VPH' \) by [7] (6.10), the homomorphism

\[
\begin{array}{ccc}
VPTG & \rightarrow & VPH \\
\downarrow & & \downarrow \\
VM & & \end{array}
\]
is an epimorphism. So the injective homomorphism 
\[ VM \rightarrow VPH' \]
is an epimorphism. Thus \( M = PH' \) by 5.5.
Q.E.D.

Note that the kernel ideal $J$ is generated by $J \cap PTG = J \cap LG$, because the generators can be written

$$\sum_{i_1 = 1}^{\frac{m_1 + 1}{2}} \sum_{i_2 = 1}^{\frac{m_2 + 1}{2}} \cdots \sum_{i_{n-1} = 1}^{\frac{m_{n-1} + 1}{2}} \sum_{A} \left[ g_{i_1, i_2', \ldots, i_{n-1}'}, g_{m_1 + 1 - i_1, i_2'', \ldots, i_{n-1}''} \right] + g_{\frac{m_1 + 1}{2}, \frac{m_2 + 1}{2}, \ldots, \frac{m_{n-1} + 1}{2}}$$

if all $m_1, m_2, \ldots, m_{n-1}$ are odd,

$$\sum_{i_1 = 1}^{\frac{m_1 + 1}{2}} \sum_{i_2 = 1}^{\frac{m_2 + 1}{2}} \cdots \sum_{i_{n-1} = 1}^{\frac{m_{n-1} + 1}{2}} \sum_{A} \left[ g_{i_1, i_2', \ldots, i_{n-1}'}, g_{m_1 + 1 - i_1, i_2'', \ldots, i_{n-1}''} \right]$$

(*)

otherwise,

where $\sum_{A}$ is the sum over all such indices

$$A = \{(i_1, i_2', \ldots, i_{n-1}'; m_1 + 1 - i_1, i_2'', \ldots, i_{n-1}'')\}$$

that satisfy: either $i_j' = i_j$ and $i_j'' = m_j + 1 - i_j$ or $i_j' = m_j + 1 - i_j$ and $i_j'' = i_j$. It is noticed that if $i_j = m_j + 1 - i_j$, then only

$$(i_1, \ldots, i_{j-1}', i_j, i_j', i_{j+1}, \ldots, i_{n-1}'; n_1 + 1 - i_1, \ldots, i_{j-1}', m_j + 1 - i_j, i_{j+1}', \ldots, i_{n-1}')$$

is in $A$ (This has to happen by general nonsense). So

$$PH' = \frac{LG}{J \cap LG'}$$

the quotient restricted Lie algebra. We need to find $PH'$.

Let $E$ be the restricted Lie subalgebra of $H'$ generated by

$$\{g_{i_1, i_2, \ldots, i_{n-1}} : 0 < i_1 \leq \ldots \leq i_{n-1}, \text{ and } i_{n-1} > 1\}.$$

Lemma 5.8 $[g_{1,1,\ldots,1}, E]$ is contained in $E$, where $[ , ]$ is the Lie product in $H'$.  

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Proof. We show by induction on \( n \) that \([g_1,\ldots,1,x]\) is in \( E \) for every basic product \( x \) of weight \( n \) in the generators \( g_{m_1,m_2,\ldots,m_{n-1}} \) of \( E \).

If \( x \) has weight 1, then \( x = g_{m_1,m_2,\ldots,m_{n-1}} \) with \( 0 < m_1 \leq \ldots \leq m_{n-1} \) and \( m_{n-1} > 1 \). Since by \((*)\)

\[
[g_1,\ldots,1,g_{m_1,\ldots,m_{n-1}}] = \sum_{i_1=1}^{m_1+1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}+1} \sum_B [g_{i_1,i_2,\ldots,i_{n-1},i_{n-1}+1-i_1,i_2,\ldots,i_{n-1}}] + g_{m_1+1,\ldots,m_{n-1}+1}^{\frac{2}{m_1+1} \cdots \frac{1}{m_{n-1}+1}} \quad m_1,\ldots,m_{n-1} \text{ are odd,}
\]

\[
\sum_B \sum_{i_1=1}^{m_1+1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}+1} [g_{i_1,i_2,\ldots,i_{n-1},i_{n-1}+1-i_1,i_2,\ldots,i_{n-1}}] \quad \text{otherwise},
\]

where \( \sum_B \) is the sum over indices \( B \) and

\[
B = A - \{(1,1,\ldots,1;m_1,m_2,\ldots,m_{n-1})\},
\]

Suppose \([g_1,\ldots,1,x]\) \( \in E \) for all \( x \) of weight \( < n \). Given \( x \in E \) of weight \( n \), we have

\[
x = [z,y] \text{ with weight } (z) < n \text{ and weight } (y) < n \text{ or } x = y^2 \text{ with weight } (y) < n. \text{ Since in the first case}
\]

\[
[g_1,\ldots,1][z,y] = [z,[g_1,\ldots,1,y]] + [y,[g_1,\ldots,1,z]],
\]

and \( y \in E \) and \( z \in E \),

\[
[g_1,\ldots,1,x] \in E.
\]

If \( x = y^2 \) with weight \( (y) < n \), since \( y \in E \) and

\[
[g_1,\ldots,1,y^2] = [y,[g_1,\ldots,1,y]],
\]

then

\[
[g_1,\ldots,1,x] \in E.
\]

Hence

\[
[g_1,\ldots,1,E] \subset E.
\]

Q.E.D.

Corollary 5.9 \( PH' \) is spanned as a vector space by \( E \) and \( g_1,\ldots,1 \).

Proof. We have \( g_1^2 = 0 \).
Q.E.D.

Let $\mathcal{P}(A)$ be the Poincaré series of the module $A$.

**Corollary 5.10**  $\mathcal{P}(H') = (1 + t^{2n-1})\mathcal{P}(VE)$.

**Proof**  From 5.5, since $g_{1,1,\ldots,1}$ has degree $2n - 1$ and $H' = VPH'$.

Q.E.D.

Let $G'$ be the vector space with basis

$$\{g_{i_1,\ldots,i_{n-1}} : 0 < i_1 \leq \ldots \leq i_{n-1}, \; i_{n-1} > 1\}$$

Let $LG'$ be the free restricted Lie algebra on $G'$. Then there are epimorphisms

$$LG' \longrightarrow E \; \text{ and } \; VLG' \longrightarrow VE.$$

**Proposition 5.11**  $E$ is the free restricted Lie algebra on $G'$.

**Proof**  Since by 4.3, every element of $TG/J$ can be written uniquely as $y + g_{1,\ldots,1}z$ with $y, \; z \in TG'$, we have

$$\mathcal{P}(H') = \mathcal{P}(TG/J) = (1 + t^{2n-1})\mathcal{P}(TG') = (1 + t^{2n-1})\mathcal{P}(VLG').$$

Since by Corollary 5.10

$$\mathcal{P}(H') = (1 + t^{2n-1})\mathcal{P}(VE),$$

we have

$$\mathcal{P}(VE) = \mathcal{P}(VLG').$$

Thus by Proposition 5.5 the epimorphisms above are isomorphisms. $E$ is free on $G'$.

Q.E.D.

We next describe the structure of the restricted Lie algebra $PH'$. Since $E = LG'$ is free, define a derivation of restricted Lie algebras

$$d : \; E \longrightarrow E$$
by

\[
dg_{m_1, \ldots, m_{n-1}} = \begin{cases} 
\sum_{i_1=1}^{m_1+1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}+1} \sum_B [g_{i_1, i_2', \ldots, i_{n-1}', \, g_{m_1+1-i_1, i_2'', \ldots, i_{n-1}''}] + g_{m_1+1, \ldots, m_{n-1}+1} \\
\frac{m_1+1}{2} \cdots \frac{m_{n-1}+1}{2} \sum_{i_1=1}^{m_1+1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}+1} [g_{i_1, i_2', \ldots, i_{n-1}', \, g_{m_1+1-i_1, i_2'', \ldots, i_{n-1}''}] 
\end{cases}
\]

\[m_1, \ldots, m_{n-1} \text{ are odd}
\]

otherwise.

and extend by linearity,

\[d[x, y] = [dx, y] + [x, dy],\]

and

\[d(x^2) = [dx, x].\]

This works because \(E\) is free.

Then \(dd\) is again a derivation, and by working in \(VLG' = TG'\), one can verify directly that \(dd \cdot g_{i_1, i_2, \ldots, i_{n-1}} = 0\), so that \(dd = 0\). Define

\[\langle g_{1, 1, \ldots, 1, x} \rangle = dx \quad \text{for} \quad x \in E.\]

Then \([g_{1, \ldots, 1}, [g_{1, \ldots, 1}, x]] = dd x = 0\) as required.

So we have

**Theorem 5.12**  \(PH'\) is spanned by \(LG' \cup \{g_{1, \ldots, 1}\}\) as a vector space. The Lie product is defined by the structure of \(LG'\) and

\[\langle g_{1, \ldots, 1}, y \rangle = dy \quad (y \in LG')\]

and \(g_{1, \ldots, 1}^2 = 0\).

**Corollary 5.13**  To obtain a set of generators of the exterior algebra \(H^*(\Omega MU(n))\), we may take any set of elements that is dual to a basis of \(PH'\).

\[\text{§6. The homology of the double loop space of } MU(n), n \geq 2.\]
Proposition 6.1  If $A$ is a Hopf algebra with basis $\{1, a\}$ and
\[ \Delta a = 1 \otimes a + a \otimes 1, \]
then $\text{Cotor}^A(K, K)$ is a polynomial algebra generated by $[a]$.

Proof  By the definition of Cotor, $\tilde{B}^* \text{ is a polynomial algebra generated by } [a]$. Since the element in $a$ is primitive, $d_1 = 0$. Thus $\text{Cotor}^A(K, K)$ is a polynomial algebra generated by $[a]$.

Q.E.D.

Proposition 6.2  If $A$ and $C$ are coalgebras over $K$, then
\[ \text{Cotor}^A(K, K) \otimes \text{Cotor}^C(K, K) = \text{Cotor}^{A \otimes C}(K, K) \]
as an algebra.

Proof  $B(A) \otimes B(C)$ is an injective resolution of $K$ by $A \otimes C$-comodules. The Künneth theorem applies.

Q.E.D.

If $X$ is a vector space, denote by $EX$ the exterior algebra on $X$, made into a Hopf algebra with $X$ primitive.

Proposition 6.3  $\text{Cotor}^{EX}(K, K) = K[[x_1], [x_2], \ldots]$, a polynomial ring, where $\{x_1, x_2, \ldots\}$ is a basis of $X$.

Proof  From 6.1, 6.2 and direct limits.

Q.E.D.

Let $H$ be any primitively generated Hopf algebra, and let $\{e_1, e_2, \ldots\}$ be an ordered basis of $PH$. Define the additive homomorphism
\[ h : EPH \rightarrow H \]
by
\[ h(e_{i_1} e_{i_2} \ldots e_{i_n}) = e_{i_1} e_{i_2} \ldots e_{i_n}, \]
where $i_1 < i_2 < \ldots < i_n$. This formula is not valid if the $e_i$ are out of order.
Lemma 6.4  If $H$ is a Hopf algebra and
\[ x = x_1 x_2 x_3 \cdots x_n \in H \]
where $x_1, x_2, \cdots x_n$ are primitive in $H$, then
\[ \Delta x = \sum_{i=0}^{n} \sum_{(i,n-i)\text{-shuffle } \sigma} x_{\sigma(1)} \cdots x_{\sigma(i)} \otimes x_{\sigma(i+1)} \cdots x_{\sigma(n)}. \]

Proof  Since $\Delta x_1 = 1 \otimes x_1 + x_1 \otimes 1$, the result holds for $n = 1$. Suppose that the result holds for $n-1$. For $x = x_1 x_2 \cdots x_n$, write
\[ z = x_1 x_2 \cdots x_{n-1}. \]
Then
\[ \Delta z = \sum_i z'_i \otimes z''_i. \]
By the definition of Hopf algebra,
\[ \Delta x = (\Delta z)(\Delta x_n) = \sum_i z'_i x_n \otimes z''_i + \sum_i z'_i \otimes z''_i x_n. \]
These are all the shuffles of $(x_1, x_2, \ldots x_n)$.

Q.E.D.

Lemma 6.5  The homomorphism $h : EPH \rightarrow H$ defined above preserves the comultiplication and is an isomorphism of coalgebras.

Proof  By Proposition 5.5, $h$ is an isomorphism. Since the $e_i$ are primitive in $EPH$ as well as in $H$, the result follows from Lemma 6.4 immediately.

Q.E.D.

Theorem 6.6  For a primitively generated Hopf algebra $H$,
\[ \text{Cotor}^H(K,K) = K[[e_1], [e_2], \ldots], \]
where $\{e_1, e_2, e_3, \ldots\}$ is an ordered basis of $PH$.
Proof Since by 6.5

\[ EPH \cong H \]

as a coalgebra and the definition of Cotor only uses the coproduct of \( H \), the result follows from 6.3.

Q.E.D.

Theorem 6.7 The spectral sequence

\[ \text{Cotor}^H_*(\Omega S^2MU(n-1)) (K, K) \implies \text{Cotor}^C_*(\Omega S^2MU(n-1)) (K, K) \]

collapses.

Proof See [8], p227 Lemma 3.8.

Lemma 6.8 The spectral sequence

\[ \text{Cotor}^H_*(\Omega MU(n)) (K, K) \implies \text{Cotor}^C_*(\Omega MU(n)) (K, K) \]

collapses.

Proof Since

\[ LG \rightarrow PH' \]

is surjective, the morphism of polynomial rings

\[ K([e_1], [e_2], \cdots) \rightarrow K([e'_1], [e'_2], \cdots) \]

is surjective, i.e. on \( E^2 \)-terms

\[ \text{Cotor}^H_*(\Omega S^2MU(n-1)) (K, K) \rightarrow \text{Cotor}^H_*(\Omega MU(n)) (K, K) \]

is surjective. Then the Lemma follows from 6.7.

Q.E.D.

Theorem 6.9 \( H_*(\Omega^2MU(n)) \) is a polynomial algebra. \( QH_*(\Omega^2MU(n)) \) has a basis isomorphic to

\[ \{[e'_1], [e'_2], \cdots\}, \]

where \( \{e'_1, e'_2, \cdots\} \) is a basis of \( PH' \) and \( \deg [e'_i] = \deg e'_i - 1 \).
**Proof**  Since the spectral sequence collapses by 6.8, so that

\[ E^\infty = \text{Cotor}^H(K, K) \]

is a polynomial algebra, lifting each generator \([e_i']\) to

\[ e_i'' \in H_*(\Omega^2 MU(n)) \]

arbitrarily, we have that \( H_*(\Omega^2 MU(n)) \) is a polynomial algebra generated by \( e_i'' \), \( i = 1, 2, \ldots \).

**Q.E.D.**
References

[10] Ted Petrie, *The Cohomology of The Loop Space of Thom Spaces*