Elementary School Mathematics Priorities

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Introduction

The February, 2006, U.S. Department of Education study, *The Toolbox Revisited*, tells us that 80% of the 1992 U.S. high school graduating class went on to college. Only about half of those students graduated with a bachelor's degree. The others dropped out. Inadequate preparation for college mathematics was a major contributor to the drop out rate. The foundation for K-12 mathematics is laid in the early years of elementary school. To succeed in college, this foundation must be solid.

A guiding principle of *No Child Left Behind* is equal opportunity for all children. Every child should learn the fundamental building blocks of mathematics. No child should be denied the preparation for high school and college mathematics that opens up the growing number of career opportunities that require mathematics.

Organization

We first describe some of the basic skills and knowledge that a solid elementary school mathematics foundation requires. We do this first briefly and then, in indented paragraphs, we elaborate. We follow our main points with a general discussion, some final comments and some suggested research topics.

Overview

Provided below is a minimal list of core concepts that must be mastered. They are the building blocks for all higher mathematics. This is not a curriculum or set of standards and it is certainly not all that students should learn in elementary school. It is also not just a list of skills to acquire. Although skills are essential in the list, understanding the concepts is also essential. This is an attempt to set priorities for emphasis in an unambiguous fashion.

The amount of high school and college level mathematics that today's workers require varies dramatically, but more and more careers are dependent on some college level mathematics. Early elementary school mathematics is the same for all students because these students should have all career options open to them. The system must not fail to offer opportunity to all students. Basics are for everyone.

It is perhaps very difficult for many fourth grade teachers to see the connection between what they need to teach and why it is necessary for the future engineer, doctor or architect. Those who regularly teach college mathematics to these students understand well what is needed from their students coming into college if they hope to fulfill the necessary mathematics requirements. A strong elementary school mathematics foundation cannot be overemphasized. In light of this, the work of our elementary school teachers is extraordinarily important.

There are five basic building blocks of elementary mathematics. Keep in mind, throughout, that mathematics is precise. There are no ambiguous statements or hidden assumptions. Definitions must be precise and are essential. Logical reasoning holds everything together and problem solving is what mathematics allows us to do. These essential building blocks are not just the foundation that algebra rests on, but, done properly, prepare the student for algebra and the mathematics beyond algebra.

Precision, lack of ambiguity and hidden assumptions, and mathematical reasoning are the fundamental defining principles of mathematics and it is difficult to adequately emphasize their importance. If a problem is not well defined with a unique set of solutions, it is not a mathematics problem. There can be no hidden assumptions in a real mathematics problem. Terms, operations, and symbols must be defined precisely. Otherwise ambiguity creeps in and we are no longer dealing with mathematics.

Although it is easy enough to say that mathematics is logical, it is more difficult to describe mathematical reasoning. Mathematical reasoning is what builds the structure of mathematics. So, in order to understand a mathematical concept, a student must absorb the mathematical reasoning that develops that concept. Dividing fractions (invert and multiply) is an important skill in mathematics but it is mathematical reasoning that explains why invert and multiply is the correct way to divide fractions. It is important that mathematical reasoning of this sort be taught and that new skills be understood through mathematical reasoning. This understanding is important and assumed throughout this discussion. Skills without understanding have little value, likewise for understanding with no skill. Each is essential.

Ultimately, solving problems is what mathematics is all about. The content of mathematics is all designed and built to solve specific types of problems. Our basic mathematics is fundamental to this enterprise because almost all other mathematics is built on it. Each new piece of mathematics allows a student to solve a new kind of problem. To be sure, at the elementary school level, some of these problems could be solved without the new mathematics being introduced, but that mathematics becomes more and more necessary as problems get more and more sophisticated. It is best to practice new mathematics on the easy problems first. It sometimes appears that learning the new mathematics in order to solve a problem is harder than just solving the problem directly, but once the newly learned mathematics feels natural it usually becomes clear that it solves the problems much more efficiently, and, importantly, can be used later to solve even harder problems that cannot be solved without the new techniques.

The five building blocks

♦**Numbers**: Numbers are the foundation of mathematics and students must learn counting and acquire instant recall of the single digit number facts for addition and multiplication (and the related facts for subtraction and division). Instant recall allows the student to concentrate on new concepts and problem solving. It is of fundamental importance in later mathematics.

This heading covers a lot of material in the earliest grades. Students must acquire some number sense. First, of course, they must learn the numbers, both to speak them and to write them. This comes along with counting and a working familiarity with and understanding of commutativity, associativity, and distributivity. Students will learn how to add (and then multiply) single digit numbers before they learn instant recall of these facts. They must have an understanding of addition, subtraction, multiplication and division that underlies the ability to instantly recall these elementary number facts. Mathematics is built level by level. Multi-digit addition and multiplication are built up from single digit operations using the place value system and the basic properties of numbers such as distributivity. The general operations reduce to the single digit number facts. Whatever their level of understanding, students without instant recall of these foundational single digit number facts are severely handicapped as they attempt to pursue the next levels of mathematics. In later courses, the student who has to quickly do the single digit computations, even if in their head, rather than just recall the answers, will find they are unable to focus completely on learning and understanding the new mathematics in their course.

♦**Place value system**: The place value system is a highly sophisticated method for writing whole numbers efficiently. It is the organizing and unifying principle for our five essential building blocks. Although its importance is often overlooked, it is the foundation of our numbering system, and, as such, deserves much more attention than it usually gets. It is much more than just hundreds, tens and ones. Arithmetic and algebra are the foundation for college level mathematics. A solid understanding of the place value system, and how it is used, is the foundation for both arithmetic and algebra. Arithmetic algorithms can only be understood in the context of the place value system. Since understanding is crucial, it begins with the place value system. Elementary school mathematics must prepare students for algebra. Working with polynomials in algebra is just a slight generalization of the place value system. The place value system is essential algebra preparation.

> The place value system is the foundation of our numbering system. The efficiency of the arithmetic algorithms are based on it. A real understanding of the basic four algorithms rests on a firm grasp of the place value system. Multiplication, for example, is little more than the combination of the place value system, distributivity, and single digit math facts for multiplication. This combination is the mathematical reasoning that makes the multiplication algorithm work.
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> The algebra of polynomials is just a generalization of the place value system. The place value system is based on 1, 10, 10 squared, 10 cubed, etc., and polynomials are based on 1, x, x squared, x cubed, etc. A solid understanding of the place value system naturally prepares students for the algebra of polynomials.
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> Without an understanding of the place value system and how it can be used there can be no real understanding of the rest of elementary school mathematics and all of the higher mathematics that rests on this. The place value system is learned in the early grades precisely because everything else depends on it so it must be taught first. Just because it is taught in the early grades does not mean that it is either simple or unimportant. On the contrary, it is a deep concept and understanding it makes all the difference. This puts a heavy burden on the teachers in these early grades and it is important that they be aware of this.

♦**Whole number operations**: Addition, subtraction, multiplication and division of whole numbers represent the basic operations of mathematics. Much of mathematics is a generalization of these operations and rests on an understanding of these procedures. They must be learned with fluency using standard algorithms. The standard algorithms are among the few deep mathematical theorems that can be taught to elementary school students. They give students power over numbers and, by learning them, give students and teachers a common language.

> The case for the importance of the standard algorithms for whole number operations cannot be overstated. They are amazingly powerful. They take the *ad hoc* out of arithmetic. They give the operations structure. The theorems that are the standard algorithms solve the age-old problem of how to do basic calculations without having to use different strategies for different numbers. They completely demystify whole number arithmetic. As an elegant, stand alone solution to an ancient problem they justify themselves.
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> There is more to the standard algorithms than just a very satisfying solution to a major problem. As students progress in their study of mathematics they will be confronted with more and more algorithms. They must start somewhere to learn about algorithms and these are the easy basic algorithms that prepare students for learning more difficult, complicated algorithms later on.

In high school and college mathematics these very same algorithms get slightly modified and generalized and used in different settings with new mathematics. This happens many times over and a mastery of the original algorithms makes this process an incremental one. The standard algorithms put all students and teachers on the same page when they make these transitions.

The standard algorithms are useful in other ways as well. The long division algorithm is probably the most important in this sense. With it, for example, it is quite easy to see that all rational numbers give rise to repeating decimals (any repeating decimal is also a rational number). It, by its very nature, also teaches estimation and begins to prepare students to understand convergence, a basic step towards calculus.

More operations than just these four come into mathematics, these are just the first four. These operations teach about operations. New operations fit into a pattern first developed with these basic four. They form a firm foundation for the conceptual development of future mathematics for the student such as the extension of these operations to rational numbers and complex numbers as well as the extension to polynomials and rational functions in algebra.

♦**Fractions and decimals**: The skills and understanding for the four basic arithmetic operations with whole numbers must be extended to fractions and decimals, and fractions and decimals must be seen as an extension of whole numbers. Students must become proficient with these operations for fractions and decimals if they are to pursue additional mathematics. Again, understanding fractions is a critical ingredient for algebra preparation. A solid grounding in fractions is a necessary prerequisite for understanding ratios, which show up everywhere including business.

Whole numbers are just not enough. Our number system must be extended to include fractions (and decimals, which are really just fractions too) in order to solve a wide variety of problems. Fractions are everywhere in mathematics and in day to day life so the ability to manipulate them with fluency is essential. They are seriously intertwined with algebra as well. First, you need them to solve simple equations like $2x=1$, and, second, in algebra, students must learn how to manipulate fractions involving polynomials, i.e. rational functions. This is, again, an incremental transition if students can operate with numerical fractions with fluency and understand and work with their definitions.

♦**Problem solving**: Single step, two step, and multi-step problems (i.e. problems that require this many steps to solve), especially word or story problems, should be taught throughout a student's mathematical education. Each new concept and skill learned can, and should be, incorporated into a series of problems of more and more complexity. The translation of words into mathematics and the skill of solving multi-step problems are crucial, elementary, forms of critical thinking. Developing critical thinking is an essential goal of mathematics education.

Mathematics is an activity. It is not enough to believe you understand something in mathematics. You must be able to *do* something with it. For example, multiplication is not *understood* if you cannot *do* it. Problem solving is what you *do* with mathematics. Problem solving at the elementary school level is a well-understood process that can be taught. Going from one step to two step to multi-step problems gradually increases the level of critical thinking.

By solving problems using new mathematics skills a student can confirm their understanding of this mathematics by doing. New skills allow students to solve problems that old skills did not suffice for. This reinforces the value of the new skills.

The difficult process of extracting a mathematics problem from a word problem requires a high level of critical thinking. However, such problems can start with great simplicity and gradually work up to immense complexity. Mathematical problem solving is a great place to hone logical critical thinking skills.

In normal daily life people are constantly being called upon to solve very complex problems that are usually not very well posed. The logical thinking and mathematical reasoning used to solve multi-step mathematics problems develops the critical thinking necessary to face life's more complex situations.

Discussion:

Students must learn the precise use of the terms, operations, and symbols of mathematics. The ability to be precise is one of the great strengths (and requirements) of mathematics and the proper use of the language of mathematics is essential in this learning process.

Mathematics is precise. The meanings of terms, operations, and symbols must be completely unambiguous or communication is lost and mathematics slips away. Students who are unsure of what they are talking about cannot hope to solve problems with such ambiguous underpinnings. For students and teachers to communicate about mathematics they must all have precise meanings for symbols and terms in common. This is easy to overlook, and often overlooked, in situations where the content seems elementary, but this is exactly when that precision should start.

In order for terms and symbols to be precise they must all have definitions. As much as arithmetic and algebra begin the content basis of mathematics, the fundamental principles of mathematics revolve around explicit definitions of terms and symbols. There are two distinct aspects to the precision of mathematics. Precision is necessary in the *doing* of mathematics and in the *communication* of mathematics.

Abstraction is part of the underlying power of mathematics and it should be taught from the earliest grades. For example, physical manipulatives are teaching aids that can help lead students to understandings in mathematics. They are not, in and of themselves, mathematics, but are teaching tools to help get to the heart of mathematics.

> One of the most important attributes of mathematics, and one that gives it much of its power, is its abstract nature. One piece of problem solving is the skill of turning complex word or story problems into concrete mathematics, in other words, to abstract the mathematics from the problem. This allows the core mathematics to be understood separately from the application, i.e. in the abstract. The benefits are manifold. The same piece of abstract mathematics can be used over and over again to solve a myriad of seemingly very different problems. Thus, an understanding of the abstract mathematics, together with the critical thinking skills necessary to extract a mathematics problem from a word problem, allows a student to solve many problems with a small skill set. Without this abstraction each new problem is exactly that: a new problem requiring special mathematics just for it. For example, it is generally possible to figure out how to multiply any two given numbers even without the standard algorithm, but the problem is different for every two given numbers. By abstracting the process to the standard algorithm for multiplication we solve the problem of multiplication for all cases once and for all.
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> Another example is college calculus. The departments, such as physics, economics and the various engineering departments, that require calculus for their majors have different applications in mind. The standard way to teach calculus for such diverse student needs is to abstract the material and teach the core principles that apply to all situations.

The place value system, fractions and the standard algorithms all contribute greatly to algebra readiness. In addition, students can learn to use variables and solve pre-algebra problems. They must also learn about graphs and graphing as functions.

> Elementary school mathematics leads up to algebra and, as such, it should prepare students for algebra. The content of our five building blocks is all necessary prerequisite for algebra, but, more than that, it prepares students for algebra because there is a natural incremental transition from arithmetic to algebra. As already mentioned, the place value system and skill (and understanding) with manipulating the algorithms and fractions is invaluable preparation. More can be done. Using abstract variables whenever possible and using commutativity, associativity, and distributivity with the variables is great preparation.
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> Reading and drawing graphs are an important component of elementary school mathematics. Using grade appropriate, but precise, definitions for functions, students should learn to think of functions as rules, not just the formulas that give the rules. This can prepare a student for algebra by avoiding confusion when

functions are more carefully defined in algebra. It can also get students used to working with formulas and equations.

Textbooks can be a tremendous help to students. With a textbook, students have the opportunity to relearn and rethink what they have seen in class. Parents can also have the opportunity to use texts to help a student at home.

> Textbooks allow students to look up concepts they are unclear on, learn from reading (a skill frequently not attained by many college students) and rereading (for most, an essential ingredient while learning mathematics). The opportunity for self paced repetition comes with a text. Likewise, parents, friends, older students, and tutors can all help with the education process if the content is made explicit in a textbook. Not having a good text available is an unnecessary and severe handicap placed on students.

Some students will not acquire all of the skills and understanding necessary to begin a formal study of algebra when the time comes. If they do not succeed at learning a fundamental third grade concept in the third grade they will struggle, and often fail, to learn more advanced concepts necessary in later grades. Such students are often completely unprepared for algebra. These students need not be lost to the subject. They are older and more mature and should be given the chance to fill in their knowledge gaps through some review or remediation process.

> For students to proceed on to algebra without the necessary background mathematics is probably pointless. If they are given appropriate attention in the class, the prepared students will suffer. If not, the unprepared students will suffer. Intervention and remediation are the appropriate responses. Given the opportunity to take the time to memorize the single digit number facts, nail down the place value system, learn to add, subtract, multiply and divide, such students have the potential to go on to learn algebra.
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> Certainly in high school there are two kinds of students, those who are still in the pipeline for a college mathematics course and those who are not. Pipeline (college preparation) algebra should not try to accommodate students without the prerequisites. A placement test is probably appropriate for pipeline algebra to be sure students meet these prerequisites. This is very much for their good. If they are allowed to take algebra and then go to college without the necessary arithmetic background, they will take a placement test in college and find themselves in remedial mathematics. (The majority of students who take a placement test in college fail it.) When such students find they are missing their mathematical foundation they tend not to be happy. They also seldom recover mathematically well enough to proceed with a college level mathematics course. Students who cannot place into pipeline algebra should have the option of proceeding with non-pipeline high school mathematics courses or getting the remediation that allows them to proceed with algebra. Every opportunity should be made to allow students

to rejoin the pipeline. The vast majority of high school graduates go on to college and the best new jobs require some mathematics.

Final Comments.

Mathematics is not a collection of unrelated topics. Mathematics is hierarchical. All of the topics discussed fit into an interconnected dependency pattern. These dependencies form a structure: the structure of mathematics. This structure is really part of the content of mathematics and was built by mathematical reasoning and it takes mathematical reasoning to make sense of the structure. The very structure of mathematics dictates that certain topics must be taught before others, and, in most cases, must be mastered before any understanding of the next topic can even be hoped for.

The core content for students who are in the pipeline for college mathematics is not in doubt. College mathematics teachers who teach students who need these courses know what is necessary. The key here is that ALL students in elementary school must be considered to be in the pipeline. Some college (or a college degree) has become a prerequisite for many of the new jobs being created in the United States. This core content for elementary school is actually quite small. For example, the top performing country internationally is Singapore. Their sixth grade textbook has a total of less than 40 pages of instruction and examples. The rest is problems. Core elementary school mathematics content is straightforward and focused. The catch is that it must be learned well to progress.

Research base

In the reading debate questions were asked like: Is it better to teach reading with phonics or without phonics? Research could answer that question. On the surface, it looks like a similar question for mathematics is: Is it better to teach mathematics with the place value system or without the place value system? However, this question makes no sense. The place value system **is** mathematics! You cannot teach mathematics without the place value system, standard algorithms and our other building blocks.

The appropriate research questions are: How is the place value system best taught? What constitutes an adequate understanding of the place value system? What is the least painful way to acquire instant recall of the single digit number facts without compromising understanding? What is the most effective way to teach the long division algorithm? Fractions are notoriously difficult for students. What approach, or collection of approaches, to fractions is most likely to succeed with almost all students?

New concepts in mathematics are presented to students each year of K-12 mathematics education. How to best teach these concepts should be thoroughly researched. Again, this differs significantly from reading where getting started is what matters most.

Postscript: Do we want domestically educated engineers?

A very high percentage of the U.S. professional science, technology, engineering, and mathematics personnel are foreign born and were given their K-12 mathematics education in their home country. If we want homegrown engineers, certain things have to take place in our K-12 mathematics education system.

If students arrive at college with large gaps in their science education they can survive, college will essentially start from scratch with science, however undesirable that may be. This is not the case with mathematics. The concepts and skills developed in every year of K-12 mathematics education are essential to success in college mathematics, mathematics that engineering students must all take. Manipulative skills with numbers and rational functions have been disparaged recently in education circles. However, the engineering student will face one class after another, year after year, where the professor comes in and writes equations on a blackboard for 50 minutes straight. Those manipulative skills must be second nature in order to survive an engineering course of instruction.

Those necessary skills and concepts for the engineering student begin with the foundation discussed in this paper in early elementary school. There is a tendency to suggest that most students do not need all of these skills because most students will not become engineers. Even if this were true, and many believe that all students actually need these skills and concepts even if they are not going to be engineers, we would be in a serious quandary. Would this mean that we should not teach them to all students? Students who don't get these skills and concepts will definitely not become engineers. So, if we want some students to be able to be engineers we have to teach these skills and concepts to these students. Is there any way to decide who in the fourth grade should be given the mathematics that would allow them to grow up to have the option of becoming an engineer? Any attempt to separate elementary school children into two groups, one group that will never have the option of becoming an engineer and another group that will be given that option, would seem grossly unfair. All elementary school children should have the option of choosing to try to be an engineer, so all children must be given the necessary mathematics in elementary school.