A Hopf ring is a (graded) ring object in the category of (graded) cocommutative coalgebras. Such an object consists, first, of a sequence \( \{H_i\} \) of abelian group objects in the category. These are better known as commutative Hopf algebras with conjugation. Being in our category they have a coproduct:

\[
\psi : H_i \longrightarrow H_i \otimes H_i.
\]

Let \( \psi(x) = \sum x' \otimes x'' \). We denote the product by \( * \). The “*” product should be thought of as “addition” in our ring as it is the pairing which gives the abelian group structure. For our ring “multiplication” we have

\[
\circ : H_i \otimes H_j \longrightarrow H_{i+j}.
\]

As with any ring, there must be a distributive law relating the multiplication and the addition. Chasing diagrams in our category we see that it is:

\[
x \circ (y * z) = \sum \pm (x' \circ y) * (x'' \circ z).
\]

Hopf rings arise naturally in the study of the \( \Omega \)-spectra associated with generalized cohomology theories. Any generalized cohomology theory, \( G^*(X) \), gives rise to a sequence of spaces, \( \{G_k\} \) with the property that \( G^k(X) \cong [X, G_k] \), the homotopy classes of maps. If \( G \) is a multiplicative theory, then \( \{G_k\} \) is a graded ring object in the homotopy category. If \( E \) represents a generalized homology theory and if there is a Künneth isomorphism for the \( E \) homology of the spaces in the \( \Omega \)-spectra for \( G \), then the sequence \( \{E_*(G_*)\} \) becomes a Hopf ring. We can thus use our understanding of generalized homologies to further our understanding of generalized cohomologies by studying their classifying spaces using Hopf rings.

There are a number of Hopf rings which have been computed. Examples are \( E_*(BP_*) \) and \( E_*(MU_*) \), \( E \) a complex orientable theory, [RW77] (the basic reference for Hopf rings); \( E_*(K(n)_*) \) and \( E_*(P(n)_*) \), \( E \) a complex orientable theory with \( I_n = 0 \), [Wil84] and [RW]; \( H_*(K(\mathbb{Z}/(p), *)) \), [Wil82, §8]; \( K(n)_*(-) \) for
Eilenberg–Mac Lane spaces, [RW80]; $K_n(k)$, [Kra90]; $H_\ast(KO)$, [Str92]; and the breakthrough description of $H_\ast(QS^0,\mathbb{Z}/(2))$ in [Tur], and its sequel for $H_\ast(QS^\ast,\mathbb{Z}/(2))$ in [ETW] followed by corresponding results for odd primes in [Li96]. Other references are [HH], [HR], [Kas94], and [KST96].

Hopf rings have a very rich algebraic structure, useful in two distinct ways: descriptive and computational. All of the above examples have their Hopf rings described with just a few generators and relations. The computations are generally carried out using Hopf ring techniques as well.

References


