A NEW RELATION ON THE STIEFEL-WHITNEY CLASSES OF SPIN MANIFOLDS

BY
W. STEPHEN WILSON

1. Introduction

Let all manifolds considered be $n$-dimensional, closed, compact, connected $C^\infty$ manifolds. Let $\tau_M : M \to BO$ be the classifying map for the stable tangent bundle of $M$. Recall that $H^*(BO)$ is a polynomial algebra on the universal Stiefel-Whitney classes, $w_1, w_2, \cdots$, where all coefficients are $\mathbb{Z}_2$. Let $S \subset H^*(BO)$, then define $I_n(S, \text{geom}) \subset H^*(BO)$ to be the ideal $\cap_M \ker \tau^*_M$ where the intersection is taken over all $n$-dimensional manifolds with $S \subset \ker \tau^*_M$. Let $H$ be an $n$-dimensional Poincaré algebra. There is a unique right-left $A$-homomorphism $\tau_H : H^*(BO) \to H$, $A$ the Steenrod algebra (see Brown-Peterson [5, Lemm 5.1, p. 44]). Define $I_n(S, \text{alg}) \subset H^*(BO)$ to be the ideal given by $\cap_H \ker \tau_H$ where $H$ runs over all $n$-dimensional Poincaré algebras such that $\tau_H(S) = 0$.

For the cases $S = \emptyset, \{w_1\}$, $I_n(S, \text{geom})$ corresponds to the intersection of $\ker \tau^*_M$ taken over all manifolds, and respectively, all oriented manifolds. For these two cases, Brown and Peterson show that $I_n(S, \text{geom}) = I_n(S, \text{alg})$ [5, Theorems 5.2 and 5.4, p. 45]. Clearly, one has $I_n(S, \text{alg}) \subset I_n(S, \text{geom})$ for all $S$. [5, p. 45] gives an example to show that equality does not always hold.

In this paper, the case where $S = \{w_1, w_2\}$ will be considered. $I_n(\{w_1, w_2\}, \text{geom})$ corresponds to the intersection of $\ker \tau^*_M$ where $M$ runs over all $n$-dimensional Spin manifolds.

Let $BO(k)$ be the $k-1$ connective covering over $BO$ and $BO(k)$ the connected $\Omega$-spectrum with $0^{th}$ term $BO(k)$. For $k = 0$ and 2, the bottom cohomology classes of $BO(0)$ and $BO(2)$ induce maps

$$\eta : H_*(X, BO(0)) \to H_*(X) \quad \text{and} \quad \gamma : H_*(X, BO(2)) \to H_{*-2}(X)$$

on the generalized homology [12]. In Section 2, the computation of $I_n(\{w_1, w_2\}, \text{geom})^\gamma$ is reduced to a problem about the image of the maps $\eta$ and $\gamma$ for the space $X = K(Z_2, n - q)$. This reduction is a generalization to Spin of Brown-Peterson results for $SO$ [5]. The major part of this paper is devoted to obtaining certain information about the image of $\eta$ for $X = K(Z_2, 2)$. These results are stated in Section 2 and used there to prove that $I_n(\{w_1, w_2\}, \text{geom})$ is not equal to $I_n(\{w_1, w_2\}, \text{alg})$ in general. In particular, it will be shown that

$$w_7 \notin I_9(\{w_1, w_2\}, \text{geom})^7 \quad \text{but} \quad w_7 \notin I_9(\{w_1, w_2\}, \text{alg})^7.$$
These results on the image of \( \eta \) give a partial computation of

\[
I_n([w_1, w_2], \text{geom})^{n-2}.
\]

The proof of the main results will be deferred until Section 5. In Section 3, generalities about \( \eta \) and \( \gamma \) are proven. These are used in Section 4 to determine \( I_n([w_1, w_2], \text{geom})^q \) for \( q = n, n - 1, \) and \( q \leq n/2 \) where it is shown to be equal to \( I_n([w_1, w_2], \text{alg})^q \). Methods for obtaining more information about higher codimensions are given in Section 6.

This paper is essentially the author’s masters thesis at M.I.T. written under the guidance of Professor F. P. Peterson. Thanks are also due to Professors Don Anderson, John Harper, and Dr. Steve Williams for many useful conversations, and Prof. Kan for his continual encouragement.

\section{I_n(Spin)}

Brown-Peterson [5] is the reference for all of Section 2.

Define \( I_n(Spin) \subset H^*(BSpin) \) to be \( \bigcap_M \ker \tau_M^* \) where the intersection is taken over all \( n \)-dimensional Spin manifolds and \( \tau_M: M \to BSpin \) is the classifying map for the stable tangent bundle.

Let \((M, f)\) represent an element in the Spin bordism group, \( \Omega_*(X, Spin) \). \( M \) is a Spin manifold and \( f: M \to X \). Define

\[
\Psi_q: H^*(BSpin) \to \Omega_*(K(Z_2, q), Spin)^*
\]

by \( \Psi_q(u)(M, f) = (\tau_M^*(u) \cdot f^*(c^q))(M) \in Z_2 \), where \( c^q \in H^q(K(Z_2, q)) \) is the generator. \( \Psi_q \) is well defined. \( H^*(BSpin) \) is a right \( A \)-module where the action is given by

\[
u \cdot a = \Phi^{-1}(\chi(a)\Phi(u)),
\]

\( \Phi: H^*(BSpin) \to H^*(MSpin) \) is the Thom isomorphism and \( \chi \) is the canonical anti-automorphism. Define

\[
\theta(X): H^*(BSpin) \otimes_A H^*(X) \to \Omega_*(X, Spin)^*
\]

by \( \theta(X)(u \otimes_A v)(M, f) = (\tau_M^*(u) \cdot f^*(v))(M) \in Z_2 \). We have then

\[
\Psi_q = \theta(K(Z_2, q)) \cdot \lambda_q
\]

where \( \lambda_q: H^*(BSpin) \to H^*(MSpin) \otimes_A H^*(K(Z_2, q)) \) is defined by

\[
\lambda_q(u) = u \otimes_A c^q.
\]

\textbf{Theorem 2.1 (Brown-Peterson, Lemma 2.2).}

\( I_n(Spin)^q = \ker (\Psi_{n-q} | H^q(BSpin)) \).

Before going further we need to recall some results on \( MSpin \) from Anderson-Brown-Peterson [3, p. 272]. Let \( J = (j_1, \ldots, j_k) \) be a sequence of integers such that \( k \geq 0 \) and \( j_i > 1 \). Let \( n(J) = \sum j_i \). Let \( MSpin \) be the Thom spectrum associated to the Spin groups. Let \( J \) be such that \( n(J) \) is even,
THE STIEFEL-WHITNEY CLASSES OF SPIN MANIFOLDS 117

and $J'$ for $n(J')$ odd. Then there are maps

$$f_\ast : \text{MSpin} \to \text{BO}(4n(J)) \quad \text{and} \quad f_{J'} : \text{MSpin} \to \text{BO}(4n(J') - 2)$$

defined in [3]. If $z \in H^n(\text{MSpin})$, let $f_\ast : \text{MSpin} \to K(Z_2, n)$ denote the corresponding spectrum map. The main result we need is:

**Theorem 2.2** (Anderson-Brown-Peterson). There is a collection of elements $z_i \in H^\ast(\text{MSpin})$ such that the map

$$F : \text{MSpin} \to \prod \text{BO}(4n(J)) \times \prod \text{BO}(4n(J') - 2) \times \prod K(Z_2, \dim z_i) = L,$$

given by $F = \prod f_\ast \times \prod f_{J'} \times \prod f_{z_i}$, induces an isomorphism on $Z_2$ cohomology.

Thus, there is a $C_2$ isomorphism

$$F(X) : H_\ast(X, \text{MSpin}) \to H_\ast(X, L)$$

where $C_2$ is the class of groups of odd order. There is also an equivalence

$$\sigma(X) : \Omega_\ast(X, \text{Spin}) \to H_\ast(X, \text{MSpin})$$

similar to the one for $SO$ in Conner and Floyd [6].

**Lemma 2.3.** There are elements

$$P \in H^{4n(J)}(B\text{Spin}), \quad Q \in H^{4n(J') - 2}(B\text{Spin}) \quad \text{and} \quad N_i = \Phi^{-1}(z_i)$$

such that $H^\ast(B\text{Spin})$, as a right $A$-module, is generated by $\{P, Q, N_i\}$ with

$$P \cdot Sq^1 A = P \cdot Sq^2 A = Q \cdot Sq^2 Sq^1 A = 0$$

as the only relations. Also, $P = \Phi^{-1}(f_\ast \alpha)$ and $Q = \Phi^{-1}(f_{J'} \alpha)$ where $\alpha$ is the bottom cohomology class of $BO(n)$ for $n = 0$ or $2 \mod 8$.

**Proof.** This follows from 2.2 and the fact that

$$H^\ast(BO(8k)) = A/A(Sq^1, Sq^2) \quad \text{and} \quad H^\ast(BO(8k + 2)) = A/A(Sq^1)$$

[9].

Define $R(X) \subset H^\ast(X) = H_\ast(X) \ast$ by $R(X) = \ker \eta^\ast$, and $S(X) \subset H^\ast(X)$ by $S(X) = \ker \gamma^\ast$.

**Lemma 2.4.** If $X$ is a CW-complex with a finite number of cells in each dimension, then the kernel of $\theta(X)$ is given by

$$\{P \otimes_A R(X), Q \otimes_A S(X)\}.$$

**Proof.** With minor modifications, the proof is the same as for Lemma 4.2 of [5, p. 47].

Let $e$ be the fundamental class of $K(Z_2, s)$. Define

$$T_s = \{a \in A \mid ae^s = 0\}, \quad R_s = \{a \in A \mid ae^s \in R(K(Z_2, s))\},$$

and

$$S_s = \{a \in A \mid ae^s \in S(K(Z_2, s))\}.$$
Let $P \subset H^*(BSpin)$ be the vector space with basis $P_j$, and $Q$ the vector space with basis $Q_j$. Define

$$F_n = \sum_{2j \geq n-i} H^i(BSpin)Sq^j.$$ 

We have:

**Theorem 2.5.** $I_n(Spin)^q = (F_n)^q + \sum_i P^i R_{n-q}^{q-i} + \sum_j Q^i S_{n-q}^{q-j}$.

*Proof.* Similar to 4.3 of [5, p. 43].

We can now state our main result and give its application. In Section 5 we prove the following:

**Theorem 2.6.** For $I$ admissible, $I = (i_1, \cdots, i_k)$, if $e(I) \geq 2$, then all $Sq^j \in R_2$ except those with $i_k = 2$. If $e(I) = 1$, and $k \geq 5$, then $Sq^j \in R_2$; if $k \leq 3$, then $Sq^j \in R_2$; for $k = 4$, not known.

**Corollary 2.7.**

$w_7 \in I_9([w_1, w_2], \text{geom})^7$ but $4 I_9([w_1, w_2], \text{alg})^7$.

*Proof.* For $J = 0$, $P_J = 1 \in H^*(BSpin)$, so by Theorem 2.5,

$$(1)Sq^4 Sq^2 Sq^1 = w_7 \in I_9(Spin)^7 \Rightarrow w_7 \in I_9([w_1, w_2], \text{geom})^7.$$ 

Now we will construct a 9-dimensional Poincaré algebra with $w_1 = w_2 = 0$ and $w_7 \neq 0$. Then, by definition,

$$w_7 \in I_9([w_1, w_2], \text{alg})^7$$

and this will conclude the proof. Let $x_2$ be a 2-dim class, and let

$$Sq^1 x_2, \quad Sq^2 Sq^1 x_2, \quad Sq^4 Sq^2 Sq^1 x_2, \quad w_4, \quad w_8, \quad \text{and} \quad w_7$$

be non-zero classes with $w_4, w_8,$ and $w_7$ the Stiefel-Whitney classes. Then the action of $A$ and the cup products are given by the following equalities:

$$1 \cdot Sq^4 Sq^2 Sq^1 x_2 = (1) Sq^4 \cdot Sq^2 Sq^1 x_2 = w_4 \cdot Sq^2 Sq^1 x_2 = w_8 \cdot Sq^1 \cdot x_2 = w_7 \cdot x_2 \neq 0.$$ 

This gives the desired Poincaré algebra.

### 3. $R_*$ and $S_*$

To study $R_*$ we will consider the map

$$\eta : H_*(K(Z_2, s), BO(0)) \to H_*(K(Z_2, s)), \quad R(K(Z_2, s)) = \ker \eta^*,$$

$$\eta^* : H_*(K(Z_2, s)) \to H_*(K(Z_2, s), BO(0))^*.$$ 

So for $a \in A$, $ac^*(\text{image} \eta) = 0$ iff $a \in R_*$. $H_*(K(Z_2, s), BO(0))$ is just the stable homotopy group of $K(Z_2, s) \wedge BO(0)$. We will study the Adams spectral sequence for this [1]. The $E_2$ term is

$$\text{Ext}_A(H_*(K(Z_2, s)) \otimes H_*(BO(0)), Z_2).$$
Now $H^*(BO(0))$ is $A/A(Sq^1, Sq^2) = A \otimes_{A_1} Z_2$ where $A_1$ is the sub-Hopf algebra generated by $Sq^1$ and $Sq^2$. So

$$E_2 = \text{Ext}_A (H^*(K(Z_2, s)), Z) \otimes (A \otimes_{A_1} Z_2), Z_2)$$

and by Anderson-Brown-Peterson [2, p. 464] or Peterson [8], we have this is isomorphic to

$$\text{Ext}_A (A \otimes_{A_1} H^*(K(Z_2, s)), Z_2).$$

By a change of rings theorem, this is $\text{Ext}_{A_1} (H^*(K(Z_2, s)), Z_2)$, from [7].

Now, in the Adams spectral sequence for $H_*(K(Z_2, s))$, $E_2 = E_\infty$ is

$$\text{Hom}_{Z_2} (H^*(K(Z_2, s)), Z_2)$$

and is in $E^{*0}$. So, since we are only interested in the image of $\eta$, we need only look at the $E^{*0}$ part of

$$\text{Ext}_{A_1} (H^*(K(Z_2, s)), Z_2)$$

which is just $\text{Hom}_{A_1} (H^*(K(Z_2, s)), Z_2)$. It induces a map

$$\eta': \text{Hom}_{A_1} (H^*(K(Z_2, s)), Z_2) \to \text{Hom}_{Z_2} (H^*(K(Z_2, s)), Z_2);$$

thus, we have:

**Proposition 3.1.** $\alpha \in A$ is in $R_*$ iff there is no $f \in \text{Hom}_{A_1} (H^*(K(Z_2, s)), Z_2)$ such that

$$\eta'(f)(\alpha^r) \neq 0 \quad \text{and} \quad d_rf = 0 \quad \text{for all} \quad r.$$

**Proposition 3.2.** If $\eta'(f)(Sq^t c^s) = 0 \forall f \Rightarrow Sq^t \in R_*$, then

$$P_J Sq^t \in I_n([w_1, w_2], \text{alg})^{n-s}$$

where $s + n(I) + 4n(J) = n$.

**Proof.**

$$\eta'(f)(Sq^t c^s) = 0 \forall f \Leftrightarrow Sq^t c^s = Sq^t \theta(c^s) + Sq^t \theta'(c^s)$$

where $\theta$ and $\theta'$ are possibly unstable operations. Then for any $n$-dimensional Poincaré algebra $H$, with

$$\tau_H([w_1, w_2]) = 0,$$

$$\tau_H(P_J Sq^t)z = \tau_H(P_J) \cdot Sq^t z = \tau_H(P_J) \cdot \theta(z) + \tau_H(P_J) \cdot \theta'(z) = 0$$

for any $z$ of dim $s$. Therefore, $\tau_H(P_J Sq^t) = 0$.

**Conjecture.** If for $Sq^t c^s$, $I$ admissible, there is an $f \in \text{Hom}_{A_1} (H^*(K(Z_2, s)), Z_2)$

such that $f(Sq^t c^s) \neq 0$, that is, $Sq^t c^s$ is not in the image of $Sq^1 + Sq^2$, then

$$P_J Sq^t \in I_n([w_1, w_2], \text{alg})^{n-s}, \quad n = n(I) + 4n(J) + s.$$

For $S_z$, the same type of analysis can be made. $E_2$ is

$$\text{Ext}_{A_1} (H^*(K(Z_2, s)) \otimes A_1/A_1(Sq^t), Z_2).$$
The corresponding propositions are:

**Proposition 3.3.** \( a \in A \) is in \( S \), if and only if there is no \( f \in E_2^{*,0} \) such that
\[
\gamma'(f)(Sq^j c^*) \neq 0 \quad \text{and} \quad d_r f = 0 \quad \text{for all} \ r.
\]

**Proposition 3.4.** If \( \gamma'(f)(Sq^j c^*) = 0 \ \forall f \), \( \Rightarrow Sq^j \in S \), then
\[
Q^j, Sq^j \in I_n([w_1, w_2], \text{alg})^{n-s}, \ n = s + n(I) + 4n(J') - 2.
\]
(Note: \( \gamma'(f)(Sq^j c^*) = 0 \ \forall f \ \Leftrightarrow \ Sq^j c^* = Sq^2 Sq^i bc^* \).

**Conjecture.** If for \( Sq^j c^* \), \( I \) admissible, there is an \( f \in E_2^{*,0} \) such that \( f(Sq^j c^*) \neq 0 \), that is, \( Sq^j c^* \) is not in the image of \( Sq^2 Sq^j \), then
\[
Q^j, Sq^j \in I_n([w_1, w_2], \text{alg})^{n-s}, \ n = s + n(I) + n(J') - 2.
\]

4. \( I_n(Spin)^q, q = n, n - 1, \) and \( q \leq n/2 \).

**Proposition 4.1.**
\[
I_n(Spin)^n = (F_n)^n = \sum_{j > 0} H^{n-j}(BSpin) Sq^j.
\]

**Proof.** This follows from 2.5.

\[
I_n([w_1, w_2], \text{geom})^n = (F_n)^n + \langle [w_1, w_2] \rangle
\]
where \( \langle [w_1, w_2] \rangle \) denotes the ideal over \( A \) generated by \( [w_1, w_2] \). So we have the following well-known result [3, p. 273]:

**Proposition 4.2.**
\[
I_n([w_1, w_2], \text{geom})^n = I_n([w_1, w_2], \text{alg})^n.
\]

**Proof.** \( F_n \subseteq I_n([w_1, w_2], \text{alg}) \) [5, p. 42].

**Theorem 4.3.** \( I_n(Spin)^q = 0 \) for \( q \leq n/2 \).

**Proof.** For dimensions \( \leq 2s, \bar{H}^*(K(Z_2, s)) \) is isomorphic to \( A \) where \( 1 \) goes to \( c^* \). \( A \) is a free \( A_1 \)-module, so \( \bar{H}^*(K(Z_2, s)) \) is free up to dim 2s. Therefore, \( E_2 \) is in \( E_2^{*,0} \) for dim \( \leq 2s \) and all differentials are zero. For \( n(I) \leq s, \) by 3.3 and 3.1, \( Sq^j = Sq^1 a + Sq^2 a' \) for \( Sq^j \in R_s \), and \( Sq^j = Sq^2 Sq^j b \) for \( Sq^j \in S_s \), \( a, a' \), and \( b \in A \), since \( n(I) \leq s \). Therefore, \( P_r Sq^j \) or \( Q_r Sq^j = 0 \). In particular, if dimension \( \{ P_r \} + n(I) \leq s, \) this is always true. So if \( n - q = s \geq n/2 \), the \( n(I) \leq n/2 \leq s \) and all terms of this type are zero. Now for \( i + j = q \leq n/2 \),
\[
(F_n)^q = \sum_{i+j \geq n} H^i(BSpin) Sq^j = \sum_{i+n/2 \geq q \geq j} H^i(BSpin) Sq^j = 0.
\]
For \( s = 1 \), the only \( Sq^j \) not in \( T_1 \) are those with \( \epsilon(I) = 1 \). So for \( Sq^j \neq Sq^1 \) (\( Sq^1 \) is handled by 4.3),
\[
Sq^j c^1 = (c^1)^{2^k} = Sq^2 Sq^1 (c^1)^{2^k-3},
\]
so by 3.2 and 3.4 we have:

**Theorem 4.4.**

\[ I_n(\{w_1, w_2\}, \text{geom})^{n-1} = I_n(\{w_1, w_2\}, \text{alg})^{n-1}. \]

5. \(R_2\)

The proof of Theorem 2.6 is divided up into the three lemmas of this section.

All \(Sq^I\) with \(e(I) > 2\) are in \(R_2\), but since \(Sq^Ic^2 = 0 \in H^*(K(Z_2, 2))\), all relations given by these \(Sq^I\) are in \(I_n(\{w_1, w_2\}, \text{alg})^{n-2}\) by 3.2. This leaves \(Sq^I\) with \(e(I) = 1\) or 2. For \(I = (i_1, \cdots, i_r, i_{r+1}, \cdots, i_k)\) with \(I\) admissible, \(i_k = 1, i_r = 2i_{r+1} + 1, e(I) = 2\), then

\[ Sq^Ic^2 = (Sq^rSq^{r+1} \cdots Sq^{i_k}c^2)^{2r-1} \]

So such an \(I\) is contained in \(R_2\) but gives rise to relations in \(I_n(\{w_1, w_2\}, \text{alg})^{n-2}\) by Theorem 3.2. The only other \(I\) with \(e(I) = 2\) are those with \(i_k = 2\).

In this case \(Sq^Ic^2 = (c^2)^{2k}\), excluding \(k = 1\) which is taken care of by 4.3. It is easily seen that this is not in the image of \(Sq^I + Sq^J\), likewise, in \(H^*(BSO)\), \(w_2^k\) is not in the image of \(Sq^1 + Sq^2\). Consider the map \(w_2 : BSO \to K(Z_2, 2)\) realizing \(w_2\). It induces a map \(w_2\) on the \(E_2\) term of the Adams spectral sequences for

\[ H_*(BSO, BO(0)) \to H_*(K(Z_2, 2), BO(0)). \]

Now since \((w_2)^{2k}\) is not in the image of \(Sq^1 + Sq^2\) there is a map

\[ f \in \text{Hom}_{A_1}(H^*(BSO), Z_2) = E_2^{*, 0} \]

with \(f((w_2)^{2k}) \neq 0\). From Anderson-Brown-Peterson [2, p. 468] or Peterson [8], we know that \(d_rf = 0 \forall r\). Therefore

\[ (w_2^r f)(c^2)^{2k} \neq 0 \quad \text{and} \quad d_r(w_2^r f) = w_2^r(d_r f) = 0 \forall r. \]

Thus \(Sq^I \in R_2\) by 3.1.

The only \(I\) left are those with \(e(I) = 1\). Summarizing, thus far we have:

**Lemma 5.1.** For \(I\) admissible \(I = (i_1, i_2, \cdots, i_k), e(I) \geq 2,\) all \(Sq^I \in R_2\) except those with \(e(I) = 2\) and \(i_k = 2\).

The main results of this section will depend on an explicit calculation of part of \(\text{Ext}_{A_1}(H^*(K(Z_2, 2)), Z_2)\). The major part of this task is to calculate the \(A_1\) structure of \(H^*(K(Z_2, 2))\) as far as we need it, which is up to about \(\dim 40\). Below \(\dim 40\), \(H^*(K(Z_2, 2))\), denoted by \(M\), is a polynomial algebra with generators in dimensions 2, 3, 5, 9, 17, and 33 which are

\[ c^2 = u_2, \quad Sq^1c^2 = u_8, \quad Sq^2Sq^1c^2 = u_8, \quad Sq^4Sq^2Sq^1c^2 = u_8, \]

etc. respectively (Serre [9]). Our approach to the problem of finding the \(A_1\)
structure of $M$ will be to break it up as an $A_1$ module into the tensor product of two $A_1$ modules $B$ and $C$.

$B$ is defined as $P(u_2, u_9, u_6) \otimes E(u_9)$ where $E$ denotes the exterior algebra and $P$ the polynomial algebra. As a $\mathbb{Z}_2$ module,

$$M = B \otimes P((u_9)^2, u_{17}, u_{33}).$$

We wish to replace $(u_9)^2, u_{17}, u_{33}$ with equivalent polynomial generators $u'_{18}, u'_{17}, u'_{33}$ such that

$$C = P(u'_{18}, u'_{17}, u'_{33}) \text{ and } M = B \otimes C$$
as $A_1$ modules. To do this, let

$$u'_{17} = u_{17} + u_6[(u_6)^2 + (u_6)^3u_2 + u_6u_6u_2]^2$$

$$u'_{18} = (u_9)^2 + Sq^1X,$$

$$u'_{33} = u_{33} + u_6(u_6)^4(u_3)^2 + u_6(u_6)^5u_2 + u_6(u_6)^6 + u_6u_6u_6 + u_6(u_6)^4u_2 + (u_6)^5u_2 + (u_6)^4u_2 + (u_6)^3u_2 + (u_6)^2u_2$$

These are obviously polynomial generators, so all that needs to be shown is that $B$ and $C$ are closed under $A_1$ action. The $A_1$ action on $M$ is described by the Cartan formula and the following relations:

$$Sq^1u_2 = u_2, \quad Sq^2u_2 = (u_2)^2, \quad Sq^3u_2 = 0,$$

$$Sq^1u_3 = u_3, \quad Sq^1u_6 = (u_6)^2,$$

$$Sq^2u_6 = Sq^1u_9 = Sq^2u_{17} = Sq^2u_{33} = 0,$$

$$Sq^1u_9 = (u_9)^2, \quad Sq^1u_{17} = (u_9)^2, \quad Sq^1u_{33} = (u_9)^2.$$

Thus, it is merely a calculation to show

$$Sq^1u'_{17} = u'_{18}, \quad Sq^1u'_{18} = 0, \quad Sq^1u'_{33} = (u'_{17})^2$$

and that $Sq^2$ is zero on $u'_{17}, u'_{18},$ and $u'_{33}$, giving us that $C$ is closed under $A_1$ action; thus, $M = B \otimes C$ as an $A_1$ module.

We now turn our attention to $B$. It is easily verified that an element of the form $u_2^4$ is not in the image of $Sq^1 + Sq^2$ and can therefore be used as an $A_1$ generator which is trivial. This fact implies that $B$ is isomorphic as an $A_1$ module to $D \otimes T$ where

$$T = P(u_2) \text{ and } D = P(u_9, u_6) \otimes E(u_4, u_2^2, u_9)$$

where when $u_2^2$ occurs in $D$ from the $A_1$ action it can be set $= 0$.

To summarize the progress thus far, we have broken $M$ up into $M = D \otimes T \otimes C$ as an $A_1$ module where $T$ is a trivial module. The only
nice property $C$ has is that it contains very few elements below dimension forty. $D$ has the property that all but a finite part of it is free. We will now show this and give the $A_1$ structure of the non-free part.

To do this, calculate the $Q_6$ and $Q_1$ homology of $D$. The $Q_6$ homology is $E(u_2^2, u_6 + u_2u_4)$. The $Q_1$ homology is $E(u_2^2, u_6 + u_4^2)$. Now, by a calculation, the low dimensional $A_1$ structure of $D$ can be determined. A part that contains these elements splits off as a direct sum from the rest of $D$ and therefore by a theorem of Wall the rest is free [11, p. 253]. We give a tabulation of the low dimensional part of $D$. $D_1$ is $A_1/A_1(Sq^3)$ with generator $u_2$. $D_2$ is $A_1/A_1(Sq^3)$ with generator $u_6 + u_2u_4 = v_1$ of dim 5 and $Z_2$ with generator $u_6 + u_2^2 + u_4u_6 = v_2$ of dim 9 with the non-trivial extension $Sq^1v_2 = Sq^2Sq^1v_1$. $D_3$ is $A_1/A_1(Sq^3)$ with generator $v_3 = u_4u_6^2 + u_6^2 + u_8$ of dim 9 and $A_1/A_1(Sq^3)$ with generator $u_6u_2 + u_6u_4^2 + u_8u_2 = v_4$ of dim 11 with non-trivial extension $Sq^2Sq^1Sq^2v_3 = Sq^2v_4$. Also we have free generators $u_2^2, u_2u_4u_2, u_2^2u_4, u_4u_6u_4$ and $u_6u_4^2$.

We now have an explicit calculation of the non-free $A_1$ structure of $D$ and $T$. We need the same for $C$. By construction this is easy. In dim 17 with generator $u_{17}^r$, we have an $A_1/A_1(Sq^2, Sq^2Sq^1)$ module $C_1$. In dim 33 with generator $u_{43}^r$, we have an $A_1/A_1(Sq^3)$ and in dim 35 with generator $u_{43}^r$ a $Z_2$ with non-trivial extension $Sq^1u_{43}^r = Sq^2Sq^1u_{43}^r$. Call this $C_2$.

Now to get the $A_1$ structure as far as we need it we must determine the non-free parts of the following modules: $C_1 \otimes D_i$ for $i = 1, 2, 3$. These are straightforward calculations. $C_1 \otimes D_1$ is $A_1/A_1(Sq^3)$ in dimensions 19 and 20, generators $v_{19}$ and $v_{20}$ with $Sq^2v_{19} = Sq^3v_{19}$. $C_1 \otimes D_2$ is the same but with generators in dimensions 28 and 29. $C_1 \otimes D_2 = A_1/A_1(Sq^2, Sq^2Sq^1)$ in dim 26.

We can now write down the entire $A_1$ structure of $M$ as far as we need it. All we need now is Ext of this. We need $Ext_{A_1}(\cdot, Z_2)$ of the following modules:

$Z_2, A_1/A_1(Sq^3), A_1/A_1(Sq^2, Sq^2Sq^1), D_2, D_3, C_1 \otimes D_1$, and $C_2$.

For $Z_2$ and $A_1/A_1(Sq^3)$, Ext is of period 8 and is as in Figures 1 and 2 where arrows denote towers. (See Anderson-Brown-Peterson [2].) Ext of $A_1/A_1(Sq^1)$ is just a tower, so Ext for $D_2$ is the sum for $A_1/A_1(Sq^1)$ and $Z_2$. For $D_3$, again Ext is the sum for $A_1/A_1(Sq^1)$ and $A_1/A_1(Sq^3)$.

For the last three cases, since the $Q_6$ homology of each is zero, we can apply a theorem of Adams that says there are no towers in Ext. We can find $E_1$ without considering the extensions and then kill off the towers. For example consider $A_1/A_1(Sq^2, Sq^2Sq^1)$ as $Z_2 + Z_2$ with non-trivial $Sq^1$. Then $E_1$ is given in Figure 3 and Ext in Figure 4, both of period 8. (See Peterson [8].) Similarly for $E_1$ and Ext for $C_1 \otimes D_1$, see Figures 5 and 6, for $C_2$, 7 and 8.

$Ext_{A_1}(M, Z_2)$ can now be written out as far as we need it. To calculate the tower killing differentials, map the entire spectral sequence into the
Adams spectral sequence for the integer homology of $K(Z, 2)$ by mapping
the bottom class of $BO(0)$ to $K(Z, 0)$. Here we can observe which elements
in $E_2$ correspond. The differentials in the $K(Z, 0)$ spectral sequence are
known (Browder [4]) and they are just the higher order Bocksteins. These
impose the same tower killing differentials in our spectral sequence.

The actual calculation gives: Towers formed by $D_2$ kill off those of $D_1$ by
a $d$. Likewise for towers formed by $D_2 \otimes u_2^{4k}$ and $D_1 \otimes u_2^{4k}$. Towers from
$D_3$ kill off those from $u_2^4$ by a $d_4$. $D_3 \otimes u_2^4$ kill $u_2$ by a $d_4$. $D_2 \otimes u_2^8$ kill $u_2^{12}$
by $d_2$ and $D_2 \otimes u_2^{12}$ kill $u_2^{16}$ by $d_6$.

Now one can write down the entire spectral sequence after the tower killing
differentials and then stare, very hard. The main lemma follows:

**Lemma 5.2.**

\[ Sq^4 Sq^2 Sq^1 \in R_2, \quad Sq^5 Sq^4 Sq^2 Sq^1 \notin R_2, \]

and

\[ Sq^6 Sq^5 Sq^4 Sq^2 Sq^1 \in R_2. \]

**Proof.** The first is easy because $E^{9,0}$ in the final version of our spectral
sequence is zero and so it follows from 3.1. For the second, we look at the
Corresponding element in $E^{17,0}$ and we see that there may be a non-trivial $d_4$
on it. If $d_4 = 0$ then it is not in $R_2$, if $d_4 \neq 0$, then it is in $R_2$. Let $c \in E^{33,0}$
coming from Ext of $C_2$ correspond to the final case. The only elements in
$E^{32,*}$ which could possibly be hit by a non-trivial differential from $c$ are
truncated towers, all of which have a non-trivial product with $\tau$ from
Ext_{A_{1}}(Z_2, Z_2). (See [2].) We can exclude the truncated tower based in $E_{32,0}$ because it is created by a $d_5$ and would have to be hit by a $d_4$, which is impossible. But note, $\tau c$ ($\tau$ is of degree $(4, 3)$) must be zero because there is no element in the Ext of $C_2$ corresponding to that dimension. So if $d_c \neq 0$ then $d_5(\tau c) = \tau (d_c) \neq 0$, but since $\tau c = 0$, $d_c$ must be zero. Thus by 3.1, we have the lemma.

If $J = \emptyset$, then $P_J = 1 \in H^{*}(BSpin)$. By 2.5 and the above lemma, we know there is a 33-dimensional Spin manifold, $M_{33}$, with

$$(1) \ Sq^{16}Sq^{8}Sq^{4}Sq^{2}Sq^{1} \neq 0.$$  

Denote the $2^i$-dimensional manifold, quaternionic projective $2^{i-2}$ space by $QP_i$. Using formula 8.1 and Lemma 2.3 of Brown-Peterson [5], which describe the right action, it is easily verified that

$$M_{33} \times QP_5 \times QP_6 \times \cdots \times QP_i, \ i \geq 5,$$

has

$$(1) \ Sq^{2i}Sq^{2i-1} \cdots Sq^{8}Sq^{4}Sq^{2}Sq^{1} \neq 0.$$  

So from Theorem 2.5 we have:

**Lemma 5.3**  
$\ Sq^{2i}Sq^{2i-1} \cdots Sq^{8}Sq^{4}Sq^{2}Sq^{1} \neq 0$ for $i \geq 4$.

This concludes the proof of Theorem 2.6.
6. Products

There is a map $\Psi : K(Z_2, i) \wedge K(Z_2, j) \to K(Z_2, i + j)$ representing the bottom cohomology class, which gives a pairing on the Adams spectral sequence (see Adams [1]) which corresponds to

$$\Psi : H_*(K(Z_2, i), BO(0)) \otimes H_*(K(Z_2, j), BO(0)) \to H_*(K(Z_2, i + j), BO(0)).$$

On the $E_2$ terms, it is a pairing

$$\Phi : \text{Ext}_{A_1}(H^*(K(Z_2, i)), Z_2) \otimes \text{Ext}_{A_1}(H^*(K(Z_2, j)), Z_2) \to \text{Ext}_{A_1}(H^*(K(Z_2, i + j)), Z_2)$$

and $d_r$ acts as a differential on products. In particular, on $E_2$, this pairing is just the map

$$\Phi : \text{Hom}_{A_1}(H^*(K(Z_2, i)), Z_2) \otimes \text{Hom}_{A_1}(H^*(K(Z_2, j)), Z_2) \to \text{Hom}_{A_1}(H^*(K(Z_2, i + j)), Z_2)$$

given by $\Phi(f \otimes g)(x) = \sum f(x') \otimes g(x'')$ where

$$\Phi^*(x) = \sum x' \otimes x''.$$

$$\Psi^* : H^*(K(Z_2, i + j)) \to H^*(K(Z_2, i)) \otimes H^*(K(Z_2, j)).$$

The use of products can give considerable information, for example: if $e(I) < n - 1$, then for

$$\Psi : K(Z_2, n - 1) \wedge K(Z_2, 1) \to K(Z_2, n)$$

we have

$$\Psi^*(Sq^r c^n) = Sq^r (c^{n-1} \otimes c^i) = Sq^r c^{n-1} \otimes c^i + \sum_{r>0} (\cdots) \otimes (c^i)^r.$$

As shown earlier, $g((c^i)^r) = 0 \forall g \in \text{Hom}_{A_1}(H^*(K(Z_2, 1)), Z_2)$. Now there is a map $g$ such that $g(c^i) \neq 0$ and $d_r g = 0 \forall r$. So in this case $d_r (f \otimes g) = \Phi((d_r f) \otimes g)$. By understanding $\Phi$ and $d_r f$ we can decide whether $Sq^r \in R_n$ or not. For example, using the above method, we have:

**Proposition 6.1.** If $e(I) \leq n - 1$ and $\text{Sq}^r \notin R_{n-1}$, then $\text{Sq}^r \notin R_n$.

For the cases $e(I) = n - 1$ and $n$, one cannot always determine if $\text{Sq}^r \in R_n$ from lower dimensional information and products.

By studying products, one can verify the following using 5.4, 2.5, 6.1, and constructing simple Poincaré algebras:

**Proposition 6.2.**

$$w_7 \notin I_n(\{w_1, w_2\}, \text{geom})^7 \quad \text{and} \quad w_7 \notin I_n(\{w_1, w_2\}, \text{alg})^7$$

for $n = 10, 11, \text{and} 12, \text{but not for} 13$, where

$$w_7 \in I_{13}(\{w_1, w_2\}, \text{geom})^7.$$
For $R$, the first two cases which cannot be decided by products are $I = (12, 6, 3, 1)$ and $(20, 10, 4, 2, 1)$.

**BIBLIOGRAPHY**


Massachusetts Institute of Technology
Cambridge, Massachusetts