PROJECTIVE DIMENSION AND BROWN–PETERSON HOMOLOGY

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(Received 1 February 1973)

$BP_*(X)$ is the reduced Brown–Peterson homology of a finite complex $X$ for a fixed prime $p$.

We study a sequence of homology theories

$$BP_*(X) \to \cdots \to BP(n+1)_*(X) \xrightarrow{\rho(n, n+1)} BP(n)_*(X) \xrightarrow{\rho(n-1, n)} BP(n-1)_*(X) \cdots \xrightarrow{\rho(-1, 0)} BP(0)_*(X).$$

Our main result states that there is an $n$ such that $\rho(n, n+1), \rho(n-1, n)$ are all epimorphisms; each of the remaining homomorphism fails to be onto; and $n$ is the projective dimension of $BP_*(X)$ as a module over the coefficient ring $BP_*$.

The first two sections are introductory in nature while the third contains the core result. The reader should be pleased to know that §4–6 are independent of each other.

§1. INTRODUCTION

Let $MU_*(\quad)$ be the reduced complex bordism theory, the homology theory associated to the unitary Thom spectrum $MU$. (This is also denoted by $\tilde{\Omega}_U(\quad)$ and $\tilde{\mathcal{M}}(\quad)$ in the literature.) This is a multiplicative homology theory with a particularly nice coefficient ring, $MU_* \cong \mathbb{Z}[y_2, y_4, \ldots, y_{2n}, \ldots]$ (dimension $y_{2n} = 2n$). In general, $MU_*(X)$ is difficult to compute; so it was natural for Conner and Smith to consider the structure of $MU_*(X)$ as a module over $MU_*[13]$. Define $\text{hom dim}_{MU_*} MU_*(X)$ to be the projective dimension of $MU_*(X)$ as a $MU_*$ module, i.e. the minimal possible length of a projective $MU_*$ resolution. Conner and Smith proved that for a finite complex $X$, the following statements are equivalent.

(i) $\text{hom dim}_{MU_*} MU_*(X) \leq 1$.

(ii) The Thom homomorphism $MU_*(X) \to H_*(X; \mathbb{Z})$ is epic.

(iii) The Thom homomorphism induces an isomorphism $Z \otimes_{MU_*} MU_*(X) \to H_*(X; \mathbb{Z})$. [13, 3.11] $k_*(\quad)$ is the connective $K$ homology theory associated to the connective $BU$ spectrum; it is a multiplicative theory with coefficient ring, $k_*(S^0) \cong \mathbb{Z}[t]$ (dimension $t = 2$). Again, we have three statements for a finite complex.

†Partially supported by NSF GP–33883.
‡Held an N.S.F. Graduate Fellowship during much of this work.
(i') \( \text{hom dim}_{MU_*} MU_*(X) \leq 2 \).

(ii') The Thom homomorphism \( MU_*(X) \to k_*(X) \) is epic.

(iii') The Thom homomorphism induces an isomorphism \( \mathbb{Z}[r] \otimes_{MU_*} MU_*(X) \to k_*(X) \).

Conner and Smith showed (i') implies (ii') and (iii') which are equivalent \([13, 11.2]\). Johnson and Smith \([19]\) then completed the analogy with the integral homology example by showing that (ii') implies (i'). Their key step was in demonstrating that the statement,

(iv) Multiplication by \( r \) in \( k_*(X) \) is monic

is equivalent to \( \text{hom dim}_{MU_*} MU_*(X) \leq 1 \).

Baas \([7]\), using ideas of Sullivan, constructed a tower of homology theories \( MU_*(\ ) \to MU_{n+1}(\ ) \to MU_{n}(\ ) \to \cdots \) such that \( MU_{n}(S^0) \cong \mathbb{Z}[y_2, \ldots, y_{2n}] \). A natural question \([6]\) was whether there exist analogs of the theorems of Conner-Smith and Johnson-Smith for these intermediate theories. We show in (4.24) that this is not possible without localization.

Recall \( \mathbb{Z}_{(p)} \) is the integers localized at the prime \( p \); i.e. the subring of rational numbers represented by fractions \( a/b \) where \( \gcd(b, p) = 1 \). (Topologists frequently denote \( \mathbb{Z}_{(p)} \) by \( Q_p \).) Quillen \([3, 27]\) described a multiplicative splitting of \( MU_*(X) \otimes \mathbb{Z}_{(p)} \) into a direct sum of shifted copies of \( BP_*(X) \), the Brown-Peterson homology of \( X \) \([9]\). \( BP_*(X) \) is a module over the coefficient ring \( BP_* = BP_*(S^0) \otimes \mathbb{Z}_{(p)} [x_1, x_2, \ldots, x_n, \ldots] \) where the dimension of \( x_n \) is \( 2p^n - 2 \). We define \( x_0 \) to be the fixed prime \( p \). By Quillen’s splitting, \( \text{hom dim}_{MU_*} MU_*(X) \otimes \mathbb{Z}_{(p)} = \text{hom dim}_{BP_*} BP_*(X) \). It is elementary that \( \text{hom dim}_{MU_*} MU_*(X) \) is the maximum of the \( \text{hom dim}_{BP_*} BP_*(X) \)'s for all primes.

For \( BP_* \), theories with the properties optimists had hoped for do exist. Using Baas’s approach to Sullivan’s theory of manifolds with singularities \([7]\), \( BP_* \) module theories \( BP_{n}(\ ) \) with \( BP_{n}(S^0) \cong \mathbb{Z}_{(p)} [x_1, \ldots, x_n] \) were constructed in \([38]\). These also fit into a tower:

\[
\begin{align*}
BP_*(\ ) & \to \cdots \to BP_{n+1}(\ ) \xrightarrow{x_n} BP_{n}(\ ) \to \cdots \to BP_{1}(\ ) \\
& \xrightarrow{\rho(0, 1)} BP_{0}(\ ) \xrightarrow{\rho(-1, 0)} BP_{-1}(\ )
\end{align*}
\]

They come equipped with exact sequences.

\[
\begin{align*}
BP_{n}(X) & \xrightarrow{x_n} BP_{n}(X) \\
& \xrightarrow{\Delta_n} BP_{n-1}(X)
\end{align*}
\]

The interlocking nature of these theories makes them a powerful tool when they are all used at once. A few of them are already familiar: \( BP_{0}(\ ) \) is homology with \( \mathbb{Z}_{(p)} \) coefficients; we define \( BP_{-1}(\ ) = H_*(\ ; \mathbb{Z}_p) \), mod \( p \) homology; and in (2.7), we prove \( BP_{1}(\ ) \) is the summand of connective \( K \)-theory localized at the prime \( p \) \([1, 5]\). Our main theorem then reads:

**Theorem 1.1.** For \( X \) a finite complex, the following conditions are equivalent.

(i) \( \text{hom dim}_{BP_*} BP_*(X) \leq n + 1 \).
(ii) $BP_\ast(X) \to BP_\ast\langle n \rangle_\ast(X)$ is epic.

(iii) The exact sequence

$$
\begin{array}{c}
BP_\ast\langle n + 1 \rangle_\ast(X) \\
\xrightarrow{\Delta_\ast^{-1}} \\
BP_\ast\langle n + 1 \rangle_\ast(X) \\
\xrightarrow{\rho(n, n - 1)} \\
BP_\ast\langle n \rangle_\ast(X)
\end{array}
$$

is short exact in that $\Delta_{n+1} = 0$.

(iv) There is an isomorphism

$$BP_\ast(X) \otimes_{BP_\ast} BP_\ast\langle n \rangle_\ast \to BP_\ast\langle n \rangle_\ast(X).$$

(v) $\text{Tor}^1_{BP_\ast}(BP_\ast(X); BP_\ast\langle n \rangle_\ast) = 0$.

(vi) There is an isomorphism

$$BP_\ast\langle n + 1 \rangle_\ast(X) \otimes_{Z(p)} [x_{n+1}] Z(p) \to BP_\ast\langle n \rangle_\ast(X).$$

(vii) $\text{Tor}^2_{BP_\ast}(BP_\ast\langle n + 1 \rangle_\ast(X); Z(p)) = 0$.

(viii) $\text{hom dim}_{MU_\ast Z(p)} MU_\ast(X) \otimes Z(p) \leq n + 1$.

The proof is given in Propositions 3.8, 3.13, and 5.12. It is based on the "splitting theorem" of [38] which states that $BP_\ast(X) \to BP_\ast\langle n \rangle_\ast(X)$ is $Z(p)$-split epic for $s < 2(p^n + p^{n-1} + \cdots + p + 1)$. This is an analog of Conner and Floyd's result that $K^0(X)$ is a direct summand of $MU^0(X)$ [10]. Although we refer often to the fundamental work of Conner and Smith [13], the completeness of our theory allows for elementary proofs which should be easily understood without prior knowledge of Conner-Smith theory.

A corollary to our main theorem and the splitting theorem is:

**Corollary 4.4.** If $X$ is a $q$ dimensional finite complex and if $q < p^n + \cdots + p + 1$, then $\text{hom dim}_{BP_\ast} BP_\ast(X) \leq n$.

This seems to be the first upper bound on the projective dimension of $BP_\ast$ (or complex bordism) modules. The theorem of Adams and Conner-Smith [11; 13] that $\text{hom dim}_{MU_\ast} MU_\ast(X)$ is finite for a finite complex is an immediate consequence of (4.4).

How does one compute lower bounds of this invariant? Conner and Smith gave a lower estimate of $\text{hom dim}_{MU_\ast} MU_\ast(X)$ which depends on the structure of possible annihilator ideals of elements of $MU_\ast(X)$. We replace this with the following ideal annihilator test.

**Proposition 4.6.** Let $X$ be a finite complex. If $x_\ast y = 0$ for $0 \neq y \in BP_\ast(X)$, then $\text{hom dim}_{BP_\ast} BP_\ast(X) \geq n + 1$.

Also Conner and Smith gave a lower bound of $\text{hom dim}_{MU_\ast} MU_\ast(X)$ involving the non-triviality of certain Steenrod operators [11]. They noted that their new test was not as effective as their annihilator ideal estimate for a stable complex

$$X(\eta, \nu) = S^0 \cup e^1 \cup_{\eta} e^3 \cup_{\nu} e^7$$

which they constructed using the first two elements of Hopf invariant one [11; 12; 14]. This motivated the following improvement of their theorem. Incidentally, our new techniques applied to the study of $X(\eta, \nu)$ show that this complex is a counterexample to the conjecture that the Conner–Smith program could be generalized directly [6].
Theorem 4.8. Let $X$ be a finite complex and let the mod $p$ Steenrod algebra, $\mathcal{A}(p)$, operate on $H_\ast(X; \mathbb{Z}_p)$ in the obvious way. Let $(Q_o)$ be the two-sided ideal of $\mathcal{A}(p)$ generated by the Bockstein. Suppose there is a selection of operations $b_1, b_2, \ldots, b_k$ in $(Q_o)$ such that the composition $b_1, b_2, \ldots, b_k$ acts nontrivially on the image of $BP\langle n \rangle_\ast(X) \rightarrow H_\ast(X; \mathbb{Z}_p)$, then $\text{hom dim}_{BP\ast}BP_\ast(X) \geq n + k + 1$ for $n = -1, 0, 1, 2, \ldots$.

Conner and Smith use two spectral sequences in their work: one of the universal coefficient type and one of the Kunneth type. We observe that these can be combined to give a spectral sequence

$$E^{2,0}_\ast(X; Y) \cong \text{Tor}^{BP\ast}(BP_\ast(X); BP\langle n \rangle_\ast(Y)) \rightarrow BP\langle n \rangle_\ast(X \wedge Y).$$

Unlike Conner and Smith, our proofs do not use new spectral sequences. For completeness, however, we have included a discussion of some of the spectral sequences which arise from the Brown–Peterson tower of spectra.

Finally, we begin a study of spherical classes in $BP_\ast(X)$. We raise some questions which do not seem to be tractable by our present methods. However, we prove the following theorem, the $n = 1$ case of which is the Stong–Hattori theorem [4; 15; 16; 17; 28; 34].

Theorem 6.1. If $X$ is a finite complex with $\text{hom dim}_{BP\ast}BP_\ast(X) \leq n$, then the Hurewicz homomorphism

$$BP_\ast(X) \rightarrow BP\langle n \rangle_\ast(BP \wedge X)$$

is a monomorphism.

Conventions. We shall work in Boardman’s stable category. In particular, we shall operate with two full subcategories. The first has as objects, pointed finite complexes; thus when we speak of a stable map of finite complexes, $f: X \rightarrow Y$, we mean a stable homotopy class represented by a base-point preserving, continuous function between suspensions of the complexes, e.g. $f: S^nX \rightarrow S^nY$. By a “connected” spectrum, we mean one which is $n$-connected for some (possibly negative) integer $n$. The second full subcategory of Boardman’s which we shall work with is the one whose objects are connected CW spectra with $Z_{(p)}$ homology of finite type. All homology and cohomology theories are reduced theories. This is no restriction, for we can add a discrete base point to a complex $X$ to form $X^+$ knowing that the reduced homology of $X^+$ is the unreduced homology of $X$. Again, we remind topologists that $Z_{(p)}$ is the integers localized at $p$ (i.e. $\mathbb{Q}_p$) and we warn algebraists that $\mathbb{Z}_p$ is the integers modulo $p$ (i.e. $\mathbb{Z}/p\mathbb{Z}$).

§2. THE BP TOWER

In [38], a tower of $BP$ module spectra was constructed using Sullivan’s theory of manifolds with singularities [7]:

$$BP = BP\langle \infty \rangle$$

$$\xrightarrow{\rho(-1, \infty)}$$

$$\xrightarrow{\rho(0, \infty)}$$

$$\xrightarrow{\rho(0, \infty)}$$

$$\cdots \rightarrow BP\langle n \rangle \xrightarrow{\rho(n - 1, \infty)} BP\langle n - 1 \rangle \rightarrow \cdots \rightarrow BP\langle 0 \rangle \xrightarrow{\rho(-1, 0)} BP\langle -1 \rangle = KZ_{(p)}$$

$$= KZ_p$$
The module structure gives a natural transformation

\[ \rho(n, \infty): BP_\ast(X) \otimes BP_\ast(Y) \to BP_\ast(X \wedge Y). \]

Letting \( X = S^0 \), \( BP_\ast(S^0) = BP_* = Z_\langle p \rangle \{ X_1, x_2, \ldots \} \) where the degree of \( x_k \) is \( 2p^k - 2 \). It is convenient to define \( x_0 \) to be \( p \), the fixed prime in consideration. Multiplication by \( x_* \) gives us an exact sequence (2.1) which we shall call the \( BP \) Bockstein exact sequence

\[ \begin{array}{c}
BP_\ast(X) \xrightarrow{x_*} BP_\ast(X) \xrightarrow{\rho(n-1,n)} BP_{\ast-1}(X) \\
\delta_n
\end{array} \quad (2.1) \]

The degree of \( \rho(n-1, n) \) is zero and the degree of \( \delta_n \) is \( -2p^n + 1 \). We have for positive \( n \), \( BP_\ast(S^0) = Z_\langle p \rangle \{ x_1, \ldots, x_n \} \), \( BP_0 = Z_\langle p \rangle \), and \( BP_{-1} = Z_p \). For \((-1)\)-connected spectra (including "honest" complexes), \( \rho(n, \infty): BP_\ast(X) \to BP_\ast(X) \) is an isomorphism for \( s < 2p^{n+1} - 2 \).

The fountainhead of our work is the following result of the second-named author.

**Theorem 2.2** [38, Corollary 5.6]. The homomorphisms \( \rho(n, \infty): BP_\ast(X) \to BP_\ast(X) \) and \( \rho(n, n+1): BP_{\ast+1}(X) \to BP_\ast(X) \) are epic for \( s < 2(p^n + \cdots + p + 1) \) and \( Z_\langle p \rangle \) split epic for \( s < 2(p^n + \cdots + p + 1) \).

For most of our purposes, we need to transform this into a statement about homology theories. In this paper, we refer to Theorem 2.2 and its corollary (2.3) as the splitting theorem.

**Corollary 2.3** For \( X \) a finite complex, the homomorphism \( \rho(n, \infty): BP_\ast(X) \to BP_\ast(X) \) and \( \rho(n, n+1): BP_{\ast+1}(X) \to BP_\ast(X) \) are \( Z_\langle p \rangle \) split epic for all but finitely many dimensions.

**Proof.** Let \( DX \) be a Spanier–Whitehead dual for \( X \). For some large \( k \) we have duality (vertical) isomorphisms in (2.4).

\[ \begin{array}{c}
BP_\ast(DX) \xrightarrow{\rho(n+1, \infty)} BP_{\ast+1}(DX) \xrightarrow{\rho(n, n+1)} BP_\ast(DX) \\
\cong \cong \cong \\
BP_{\ast-1}(X) \xrightarrow{\rho(n+1, \infty)} BP_{\ast+1}(DX) \xrightarrow{\rho(n, n+1)} BP_\ast(DX) \\
\cong \cong \cong \\
BP_{\ast-1}(X) \xrightarrow{\rho(n+1, \infty)} BP_{\ast+1}(X) \xrightarrow{\rho(n, n+1)} BP_{\ast-1}(X). \\
\end{array} \quad (2.4) \]

For low \( s \), the top horizontal homomorphisms are \( Z_\langle p \rangle \) split epic by Theorem 2.2.

Let \( \mathscr{A}(p) \) be the mod \( p \) Steenrod algebra and let \( Q_i \) be the Milnor elements [23].

**Lemma 2.5** [38, Proposition 1.7]. \( H^\ast(BP_\ast, Z_p) = \mathscr{A}(p)/\mathscr{A}(p)(Q_0, \ldots, Q_n) \) and if \( \Delta_n: BP_{\ast+1}(X) \to S^2 p^{n-1} BP_\ast \) is the map of spectra induced by (2.1), then \( \Delta_n^\ast(1) = \lambda Q_n, \) \( 0 \neq \lambda \in Z_p \).

**Remark 2.6.** Smith [30] and Toda [35] have studied the existence of stable complexes \( V(n) \) satisfying the property that \( BP_\ast(V(n)) \cong BP_\ast(p, x_1, \ldots, x_n) \) or, equivalently, \( H^\ast(V(n); Z_p) \cong E(Q_0, Q_1, \ldots, Q_n) \). A third equivalent property is that \( \pi_\ast(BP_\ast(X) \wedge V(n)) \cong BP_\ast(X \wedge V(n)) \cong Z_p \). So we can think of the \( V(n) \) existence problem as a factorization.
problem. For a given prime \( p \), for which \( n \) can we factor the \( Z_p \) Eilenberg–MacLane spectrum as \( KZ_p \cong BP(n) \wedge V(n) \)?

The theories \( BP(n) \) are new and unfamiliar; however, \( BP(1) \) has a familiar interpretation. It is a folk theorem that the spectrum for connective \( K \)-theory splits into \( p - 1 \) many shifted copies of a ring spectrum \( G \) when localized at \( p \). \( G_\ast(S^0) = G = Z_{(p)}[y] \) where the degree of \( y \) is \( 2p - 2 \). We shall show that \( G \) and \( BP(1) \) are homotopy equivalent. We need this identification so that the results of Section 6 give the Strong–Hattori theorem. We shall not prove the whole of the folk theorem, just enough to get a nice grip on \( G \). Let \( bu(p) \) denote the spectrum for connective \( K \)-theory localized at \( p \).

**Proposition 2.7**

(i). There is a ring spectrum \( G \) and a map of ring spectra \( G \to bu(p) \).

(ii) \( G_\ast(-) \to bu(p)_\ast(-) \) maps isomorphically to a direct summand.

(iii) There is a homotopy equivalence \( G \cong BP(1) \).

**Proof.** We need the following results from the Adams, Anderson–Meiselman splitting of \( K \)-theory [1; 5]. \( K \)-theory localized at \( p \) has a natural, multiplicative, representable direct summand, \( K_0(-) \). (Notational warning: the subscript refers to the zero coset in \( Z_p = Z/pZ \) not to a dimension. See [1, pp. 90–92].) There is a periodicity isomorphism (2.8) given by the external product with a generator of \( K_0(S^{2p-2}) \)

\[
\phi : K_0(X) \to K_0(S^{2p-2} \wedge X).
\]

\[
K_0(S^p) = \begin{cases} Z(p) & \text{if } n = 2i(p-1) \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( G(0) \) be the classifying space for the representable functor \( K_0(-) \). The natural direct summand statement says that there is an equivalence \( Z_{(p)} \times BU_{(p)} \cong G(0) \times Y \) for some \( Y \). Let \( G(n) \) be the \( 2n(p-1) - 1 \) connected covering of \( G(0) \).

**Claim 2.10.** There is an equivalence \( \Omega^{2(p-1)}G(n+1) \cong G(n) \).

**Proof of claim.** \( \Omega^{2(p-1)}G(0) \cong G(0) \) by (2.8). \( G(0) \cong G(1) \times Z_{(p)} \), so \( \Omega^{2(p-1)}G(1) \cong G(0) \). By (2.9) and the definition of the \( G(n) \)'s we have a fibration, \( G(n+1) \to G(n) \to K(Z_{(p)}, 2n(p-1)) \). Applying the functor \( \Omega^{2(p-1)} \), we get \( G(n) = \Omega^{2(p-1)}G(n+1) \to G(n-1) \to K(Z_{(p)}, 2(n-1)(p-1)) \) by induction.

The property (2.10) allows the spaces \( \{G(n)\} \) to define an omega spectrum \( G \). (Note that the \( 2n(p-1) \) st space of \( G \) is \( G(n) \).) Similarly, we can represent \( bu_{(p)} = \{bu_{(p)}(n)\} \), where \( bu_{(p)}(n) \) is the \( 2n(p-1) - 1 \) st connected covering of \( Z_{(p)} \times BU_{(p)} \). (Compare [2].) Again \( \Omega^{2(p-1)}bu_{(p)}(n) \cong bu_{(p)}(n-1) \) by the same proof. Now by our construction and the splitting of \( Z_{(p)} \times BU_{(p)} \), we get \( bu_{(p)}(n) \cong G(n) \times W(n) \), some \( W(n) \), giving us (ii).

\[
\begin{align*}
G(m) \wedge G(n) & \to G(m+n) \\
& \downarrow \\
bu_{(p)}(m) \wedge bu_{(p)}(n) & \to bu_{(p)}(m+n) \\
& \downarrow \\
G(0) \wedge G(0) & \to G(0) \\
& \downarrow \\
bu_{(p)}(0) \wedge bu_{(p)}(0) & \to bu_{(p)}(0)
\end{align*}
\]

(2.11)
To prove (i), we need to define the dashed arrows in (2.11) to make the diagram homotopy commute. The bottom square of (2.11) is given by the fact that $K_0(-)$ is a multiplicative summand of $K$-theory. $G(m) \wedge G(n)$ is $2(n + m)(p - 1) - 1$ connected so the composition $G(m) \wedge G(n) \rightarrow G(0) \wedge G(0) \rightarrow G(0)$ lifts uniquely to $G(m + n)$. The maps for $bu_{(p)}$ are defined similarly. The top square homotopy commutes since both compositions are the unique lifts of the composition,

$$G(m) \wedge G(n) \rightarrow G(0) \wedge G(0) \rightarrow bu_{(p)}(0) \wedge bu_{(p)}(0) \rightarrow bu_{(p)}(0).$$

We can now move on to (iii). Let $BP\langle 1 \rangle(n)$ be the classifying space for the functor $BP\langle 1 \rangle 2n(p-1)(-)$. From the exact sequence (2.1) and the fact that $BP\langle 0 \rangle = KZ_{(p)}$, we see that $\Omega^{2(p-1)}BP\langle 1 \rangle(n + 1) \simeq BP\langle 1 \rangle(n)$ and that $BP\langle 1 \rangle(n)$ may be taken to be the $2n(p - 1) - 1$ connective covering of $BP\langle 0 \rangle$. In [25; 38], $BP\langle 1 \rangle(0)$ and $G(0)$ are shown to be homotopy equivalent. Thus $BP\langle 1 \rangle(n) \simeq G(n)$ as spaces and $BP\langle 1 \rangle \simeq G$ as spectra.

We want our map $\rho(1, \infty): BP \rightarrow BP\langle 1 \rangle$ to be equivalent to the standard map $\rho: BP \rightarrow G$. The next proposition will suffice.

**Proposition 2.12.** There exists a homotopy equivalence $f$ making diagram (2.13) homotopy commute.

$$\begin{array}{ccc}
BP & \xrightarrow{f} & G \\
\downarrow & \simeq & \downarrow \\
BP & \xrightarrow{\rho(1, \infty)} & BP\langle 1 \rangle
\end{array} \quad (2.13)$$

**Proof.** $\rho(1, \infty): E_2^{*,*}(BP) \cong H^*(BP; BP^*) \rightarrow E_2^{*,*}(BP) \cong H^*(BP; BP^*)$ is epic. The two spectral sequences $E_2^{*,*}(BP) \Rightarrow BP^*(BP)$ and $E_2^{*,*}(BP) \Rightarrow BP\langle 1 \rangle^*(BP)$ collapse and so $\rho(1, \infty): BP^*(BP) \rightarrow BP\langle 1 \rangle^*(BP)$ is onto. Thus there is an $f$ such that $[\rho(1, \infty) \circ f] = \rho(1, \infty) [f] = [\rho \circ \rho]$. Now $f$ must induce an isomorphism on $\pi_0(BP)$ and so $f^*: H^*(BP; Z_p) \rightarrow H^*(BP; Z_p) = \mathcal{A}(p)/(Q_0)$ is an isomorphism. Since $BP$ is a localized spectrum, this implies $f$ is a homotopy equivalence.

For certain constructions, we need the module $BP_{\bullet}(X)$ to be finitely generated over $BP_{\bullet}$ when $X$ is a finite complex. Originally, this was proved by Novikov and Smith independently [24; 32] using the properties of coherent rings. It is of interest that this work is independent of coherence; so we shall point out a more recent proof. Quillen [26] showed that for a connected complex of dimension $k$ the $MU_{\bullet}$ generators of $MU_{\bullet}(X)$ are in dimensions $\leq 2k$. $MU_{\bullet}(X)$ is a finitely generated abelian group in dimensions $\leq 2k$; so Quillen's theorem gives the finite generation result. A new proof of Quillen's theorem using (2.2) can be found in [38]. The same proof shows that the $Z_{(p)} [X_1, \ldots, X_k] \subset BP_{\bullet}$ generators of $BP^*(n)_{\bullet}(X)$ are in dimensions $\leq 2k$. For $BP\langle 1 \rangle^\bullet_{\bullet}(X)$, the generators are known to be in dimensions $\leq k$ [33].

**Question 2.14.** Is it possible to give a more precise statement about the generators which depends on hom dim $BP_{\bullet}BP_{\bullet}(X)$?
§3. TORSION AND PROJECTIVE DIMENSION

In this section, we prove perhaps the most interesting of the equivalences of the main theorem. We begin with a review of standard facts about localization and torsion (e.g. see [8]).

Fix a graded, integral domain with unit, $R_*$. It is convenient to assume that $R_*$ is concentrated in even dimensions. Let $T \subseteq R_*(0)$ be a set containing 1 which is closed under multiplication (such is called a multiplicative set). Given a graded $R_*$ module $M_*$, we define $T^{-1}M_*$ to be the set of equivalence classes of "fractions" $m/t$, $m \in M_*$ and $t \in T$, under the relation induced by:

$$m/t \sim m'/t' \iff \exists \ t_0 \in T \ \forall \ t_0' \ t_0tm = t_0tm'.$$

Abusively, we let $m/t$ stand for the equivalence class of $m/t$. $T^{-1}M_*$ is an $R_*$ module by the rules for addition and multiplication of fractions recalled from elementary school. A homomorphism of graded $R_*$ modules (of arbitrary degree), $f: M_* \to N_*$; induces a homomorphism $T^{-1}f: T^{-1}M_* \to T^{-1}N_*$ by the rule $(T^{-1}f)(m/t) = f(m)/t$. We see that $T^{-1}$ is an endofunctor of the category of graded $R_*$ modules and homomorphisms of arbitrary degree. There is a canonical homomorphism of graded $R_*$ modules $\tau: M_* \to T^{-1}M_*$ defined by $\tau(m) = m/1$. We call it the localization homomorphism; it induces a natural equivalence of functors, $\tau: T^{-1}R_* \otimes R_*, M_* \to T^{-1}M_*$ defined on generators by $\tau(r/t \otimes m) = (rm)/t$. $T^{-1}$ is an exact functor and $T^{-1}R_*$ is a flat $R_*$ module. (Right exactness follows from the equivalence $\tau$; left exactness is checked easily.) There is a category of $R_*$ module homology theories: the objects are homology theories $E_*(-)$ with nice natural pairings $R_* \otimes E_*(-) \to E_*(-)$ and the morphisms are natural transformations of homology theories (of arbitrary degree) respecting these natural pairings. The exactness of $T^{-1}$ means that it induces an endofunctor of this category of $R_*$ module homology theories. We call this induced functor $T^{-1}$ also.

Where there is localization, there is torsion. Fix a multiplicative subset $T \subseteq R_*(0)$. An element of the kernel of the localization homomorphism $\tau: M_* \to T^{-1}M_*$ is said to be a $T$-torsion element of the graded $R_*$ module $M_*$. If $\tau(M_*) = 0$ (or equivalently, $T^{-1}M_* = 0$), we say $M_*$ is a $T$-torsion module. On the other hand, if $\tau$ is a monomorphism, we say $M_*$ is $T$-torsion free. If $f: M_* \to N_*$ is a homomorphism of graded $R_*$ modules, $f$ is monic modulo $T$ torsion (epic modulo $T$ torsion; zero modulo $T$ torsion) provided that $T^{-1}f$ is a monomorphism (epimorphism; zero morphism).

A motivating example concerns complex connective $K$-theory, $k_*(X)$, which is a module homology theory over the coefficient ring $k_*(S^0) \cong Z[t]$. $T = \{1, t, t^2, \cdots\} \subset Z[t]$, $0$ is a multiplicative set. $T^{-1}k_*(X)$ may be identified with $K_*(X)$, complex $K$-homology theory (Notice that localization unconnects a connected homology theory when positive dimensional elements are present in $T$.) We now interpret the key result of [19] as saying: $k_*(X)$ is $T$-torsion free if and only if the projective dimension of $MU_*(X)$ as a $MU_*$ module is at most one.

Our attention will center on the case when $R_* = BP_* \cong Z(p)[x_1, x_2, \cdots]$. Define $T_\infty$ to be the multiplicative subset $\{1, x_\infty, x_\infty^2, \cdots\} \subset BP_*$. As a convention, we let $x_0$ denote $p$ so
that $T_0 = \{1, p, p^2, \cdots \}$ and so that $T_0^{-1}$ is equivalent to the functor $- \otimes Q$. Note that an element $y$ of a $BP_\ast$ module $M_\ast$ is a $T_\ast$ torsion element if and only if $x_\ast^s y = 0$ for some power $s$. A $BP_\ast$ module $M_\ast$ is $T_\ast$ torsion free if and only if multiplication by $x_\ast$ in $M_\ast$ is a monomorphism.

**Lemma 3.1.** Let $\varphi_\ast: E_\ast(X)$ be a natural transformation of $R_\ast$ module homology theories defined on Boardman’s stable category [36] and let $T \subseteq R_\ast([0])$ be a multiplicative subset. If $\varphi_\ast$ is monic modulo $T$ torsion (epic modulo $T$ torsion; modulo $T$ torsion) for all finite CW complexes, then $\varphi_\ast$ is monic modulo $T$ torsion (epic modulo $T$ torsion; zero modulo $T$ torsion) for all CW spectra in Boardman’s category.

**Proof.** $E_\ast(X) \cong \lim E_\ast(X_\ast)$ and $F_\ast(X) \cong \lim F_\ast(X_\ast)$ where the limits are taken over all finite subspectra of the spectrum $X$ [36, 12.9]. $\lim$ is an exact functor which commutes with $T^{-1}$.

**Proposition 3.2.** Let $X$ be a connected CW spectrum and let $0 \leq k \leq n$. In the $BP$--Bockstein sequence (2.1), $\Delta_n$ is zero modulo $T_k$ torsion (i.e. $T_k^{-1} \Delta_n = 0$).

**Proof.** By Lemma 3.1, it suffices to prove the proposition for the case when $X$ is finite. For $k = 0$, the result follows the fact that $\Delta_n = 0$ when $X = S^0$ and from the usual Serre modulo-torsion theory. For $k = n$, it is an immediate consequence of the exactness of the sequence (2.1). Now suppose $1 \leq k \leq n - 1$ and suppose $(T_k^{-1} \Delta_n)(y/x_k^s) \neq 0$ for some element $y/x_k^s \in T_k^{-1}BP(n-1)_\ast(X)$ where $X$ is a finite complex. Multiplication by $x_k$ is an isomorphism in $T_k^{-1}BP(n)_\ast(X)$; so $0 \neq (x_k^s)(T_k^{-1} \Delta_n)(y/x_k^s) = (T_k^{-1} \Delta_n)(x_k^s y/x_k^s) = T_k^{-1} \Delta_n(x_k^{-s} y/1)$ for all non-negative $t$. This implies $\Delta_n$ is non-zero for infinitely many dimensions contradicting the splitting theorem (2.3). So $T_k^{-1} \Delta_n \equiv 0$.

**Proposition 3.3.** Let $X$ be a connected CW spectrum and let $0 \leq k \leq n$. If $BP(n)_\ast(X)$ is $T_k$ torsion free, then $BP(n+1)_\ast(X)$ is also $T_k$ torsion free.

**Proof.** Let $\tau'$ be the composition $BP(n)_\ast(X) \to BP(n)_\ast(X) \to T_k^{-1}BP(n)_\ast(X)$ where $\tau$ is the localization homomorphism which is a monomorphism by the hypothesis. Since $T_k^{-1} \Delta_{n+1} \equiv 0$ (3.2), the bottom row in diagram (3.4) is short exact. The top row is exact and the diagram (of abelian groups) commutes.

$$
\begin{array}{cccc}
BP(n+1)_\ast & \xrightarrow{\Delta_{n+1}} & BP(n+1)_\ast & \xrightarrow{\tau} & BP(n)_\ast(X) \\
\downarrow & & \downarrow & & \downarrow \\
T_k^{-1}BP(n+1)_\ast(X) & \to & T_k^{-1}BP(n+1)_\ast(X) & \to & T_k^{-1}BP(n)_\ast(X)
\end{array}
$$

By an induction on dimension, $\tau_{s-2p+1+2}$ is monic. (We need the hypothesis that $X$ is connected to start induction.) By the strong “five” lemma, $\tau_s$ is monic. This being true for all $s$, the localization homomorphism $\tau: BP(n+1)_\ast(X) \to T_k^{-1}BP(n+1)_\ast(X)$ is then a monomorphism.

**Corollary 3.5.** Let $X$ be a connected CW spectrum and let $0 \leq k \leq n \leq m$. If $BP(n)_\ast(X)$ is $T_k$ torsion free, then $BP(m)_\ast(X)$ and $BP_\ast(X)$ are $T_k$ torsion free.
Proof. Use the obvious induction to show $BP(m)_*(X)$ is $T_k$ torsion free. Observe that $BP_*(X) = BP(n)_*(X)$ for some large $m$.

**Corollary 3.6.** Let $X$ be a connected CW spectrum. If $BP(n)_*(X)$ is $T_n$ torsion free, then $BP(n+1)_*(X)$ is $T_{n+1}$ torsion free. (If $\Delta_n \equiv 0$, then $\Delta_{n+1} \equiv 0$ or if $\rho(n-1, n)$ is epic, then $\rho(n, n+1)$ is epic.)

**Proof.** $BP(n+1)_*(X)$ is $T_n$ torsion free (3.3); so $\Delta_{n+1} \equiv 0$ in the $BP$ Bockstein sequence (3.2) implying multiplication by $x_{n+1}$ is monic in $BP(n+1)_*(X)$.

**Corollary 3.7.** Let $X$ be a connected CW spectrum and let $0 \leq k \leq n$. $\rho(n, \infty) : BP_k(X) \to BP(n)_*(X)$ is epic modulo $T_k$ torsion. Furthermore, if $BP(n+1)_*(X)$ is $T_k$ torsion free, then $\rho(n, \infty)$ is epic.

**Proof.** Implicit in the splitting theorem (2.3) is the fact that for a finite complex $X$, there is an integer $m$ such that $\rho(m, \infty) : BP_k(X) \to BP(m)_*(X)$ is epic (namely an $m$ such that a Spanier-Whitehead dual of $X$ has dimension less than $2(p^n + \cdots + p + 1)$.) So the first statement for a finite complex follows from the fact that $\rho(m-1, m), \ldots, \rho(n, n+1)$ are epic modulo $T_k$ torsion (3.2). This is generalized to CW spectra by (3.1). If $BP(n+1)_*(X)$ is $T_k$ torsion free, $\Delta_{n+1} \equiv 0$ (3.2) and $\rho(n, n+1)$ is epic. By iteration of Corollary 3.6, $\rho(n, n+1), \rho(n+1, n+2), \cdots$ are epic.

**Proposition 3.8.** For a connected CW spectrum $X$, the following three conditions are equivalent:

(i) $\rho(n, \infty) : BP_k(X) \to BP(n)_*(X)$ is epic;

(ii) $\rho(n+1, n) : BP(n+1)_*(X) \to BP(n)_*(X)$ is epic;

(iii) $BP(n+1)_*(X)$ is $T_{n+1}$ torsion free.

**Proof.** The equivalence of (ii) and (iii) follows from the $BP$ Bockstein exact sequence (2.1). The implication (iii) $\Rightarrow$ (i) is in (3.7). Since $\rho(n, \infty) = \rho(n, n+1) \circ \rho(n+1, \infty)$, (i) implies (ii).

As Larry Smith remarked [31] on the analogous theorem relating integral homology, connective $K$-theory, and complex bordism, this proposition gives a curious phenomenon. Thinking cohomologically and letting $X$ be a finite complex, this proposition says that if every stable map $f : X \to BP(n)$ lifts through $\rho(n, n+1)$ to $f''$ (in diagram (3.9), then every such $f$ lifts through $\rho(n, \infty)$ to $f''$.

![Diagram](3.9)
The modifier every is essential. It is easy to construct examples of complexes $X$ such that there is an $x \in H_*(X; Z_p)$ which is the mod $p$ reduction of an integral homology class which is not Steenrod representable (not in the image of the Thom homomorphism $MU_*(X) \to H_*(X; Z)$.)

So far in this section, our results have been corollaries to the splitting theorem. Now to prove results about projective dimension, we need more data. Our proofs tend to be inductive; so the following definition of hom dimension, $\text{hom dim}_{BP_*} M_*$, $M_*$ a graded $BP_*$ module, suits our purposes. $\text{hom dim}_{BP_*} M_* = 0$ if and only if $M_*$ is $BP_*$ projective. If $M_*$ is not $BP_*$ projective, let $0 \to N_* \to P_* \to M_* \to 0$ be a short exact sequence of graded $BP_*$ modules with $P_*$ $BP_*$ projective. We inductively define $\text{hom dim}_{BP_*} M_* = 1 + \text{hom dim}_{BP_*} N_*$. Our inductive step will use a geometric resolution theorem which was proved by Conner and Smith using the basic idea of Atiyah as adapted by Landweber. To begin induction, we need the purely algebraic fact that graded projective $\mathbb{Z}_p[x_1, x_2, \ldots]$ modules are free [13, 3.2]; we omit its proof. Also we need the theorem of Conner and Smith that $BP_*(X)$ being a free (= projective) $BP_*$ module is equivalent to $H_*(X; Z_{(p)})$ begin $\mathbb{Z}_p$ free. We feel the reader will appreciate our inclusion of sketches of Conner and Smith's proof of this last result and of the resolution theorem.

**Proposition 3.10 (Conner-Smith).** Let $X$ be a connected $CW$ spectrum with $H_*(X; Z_{(p)})$ of finite type. $H_*(X; Z_{(p)})$ is free $\mathbb{Z}_p$ if and only if $BP_*(X)$ is $BP_*$ free.

**Sketch proof.** If $H_*(X; Z_{(p)})$ is $\mathbb{Z}_p$ free, the spectral sequence $E^2_{*,*}(X) \Rightarrow H_*(X; BP_*)$ collapses as the differentials are torsion valued. Thus the associated graded object $E^2_{*,*}(X) = E^2_{*,*}(X) \cong H_*(X; Z_{(p)}) \otimes BP_*$ to the filtration of $BP_*(X)$ is $BP_*$ free implying $BP_*(X)$ is $BP_*$ free. Now assume $BP_*(X)$ is $BP_*$ free. If $H_*(X; Z_{(p)})$ has torsion, it is easy to show using the above spectral sequence that the lowest dimensional torsion element $y$ is in the image of $\rho(0, \infty): BP_*(X) \to H_*(X; Z_{(p)})$ which factors through $\rho(0, \infty): BP_*(X) \otimes_{BP_*} \mathbb{Z}_{(p)} \to H_*(X; Z_{(p)})$. So $\rho(0, \infty)(z) = y$ for some $z \in BP_*(X) \otimes_{BP_*} \mathbb{Z}_{(p)}$ which is a torsion free group. We perceive a contradiction when we note that $\rho(0, \infty) \otimes Q$ is an isomorphism, $z \otimes 1 \neq 0$, and $y \otimes 1 = 0$.

**Definition 3.11.** For $-1 \leq n \leq \infty$, a (geometric) $BP\langle n \rangle_*$ resolution of a connected $CW$ spectrum $X$ is a stable cofibration of connected $CW$ spectra $W \xrightarrow{q} A \xrightarrow{i} X$ such that $H_*(A; \mathbb{Z})$ is free abelian and the induced sequence

$$0 \to BP\langle n \rangle_*(W) \xrightarrow{q_*} BP\langle n \rangle_*(A) \xrightarrow{i_*} BP\langle n \rangle_*(X) \to 0$$

is short exact.

**Proposition 3.12 (Conner-Smith).** Every connected CW spectrum has a $BP_\infty = BP\langle \infty \rangle_*$ resolution.

**Sketch proof for a finite complex.** (The extension to connected CW spectra is given in [20] and is credited to Smith.) Given a finite complex $X$, we choose a Spanier–Whitehead
dual $DX. MU^*(DX)$ is finitely generated over $MU^*$, say by $f_j: DX \to S^nMU$. Let $(S^nMU)^{k_i}$ be the $k_i$ skeleton of $S^nMU$ where the $k_i \geq$ dimension of $DX$. Then $H^*((S^nMU)^{k_i};Z)$ is free abelian and $(\vee f_j)^*: MU^*((S^nMU)^{k_i};Z) \to MU^*(DX;Z)$ is epic. Let $A$ be a dual of $(\vee(S^nMU)^{k_i})$; so we get a stable cofibration $W \xrightarrow{g} A \xrightarrow{f} X$ with $H_*(A;Z)$ free abelian and $MU_*(f)$ epic. Thus $BP_*(f)$ is epic as required.

**Proposition 3.13.** For a connected CW spectrum $X$ with $H_*(X;Z_{(p)})$ of finite type, the following statements are equivalent, $-1 \leq n$.

1. hom dim_{BP_*}BP_*(X) \leq n + 1.
2. $\rho(n, \infty): BP_*(X) \to BP(n)_*(X)$ is epic.
3. $BP(n + 1)_*(X)$ is torsion free.

**Proof.** The equivalence of (ii) and (iii) was noted in (3.8). The proof is by induction on $n$. Proposition 3.10 gives the $n = -1$ case. Use Proposition 3.12 to choose a $BP_*$ resolution of $X$ inducing the commutative diagram (3.14) with exact rows.

![Diagram](image)

Note that since $H_*(A;Z)$ is free abelian, $BP(n)_*(A)$ is $T_n$ torsion free and $\rho_A = \rho(n, \infty)$ is epic (3.6 and 3.7). $\rho_X$ is epic modulo $T_n$ torsion free (3.7); so by commutativity $\rho_2$ is epic modulo $T_n$ torsion and by exactness $\rho_2$ is epic modulo $T_n$ torsion. We complete the proof:

$\rho_X$ is epic $\iff f_2$ is epic $\iff g_2$ is monic $\iff BP(n)_*(W)$ is $T_n$ torsion free $\iff$ hom dim_{BP_*}BP_*(W) $\leq n \iff$ hom dim_{BP_*}BP_*(X) $\leq n + 1$.

The next-to-the-last equivalence was by induction.

**Corollary 3.15.** Let $X$ be a connected CW spectrum with $H_*(X;Z_{(p)})$ of finite type. $X$ has a $BP_*$ resolution if and only if hom dim_{BP_*}BP_*(X) $\leq n + 1$.

**Proof.** Consider diagram (3.14). If hom dim_{BP_*}BP_*(X) $\leq n + 1$, $\rho_X$ is epic (3.13) and thus $f_2 = BP(n)_*(f)$ is epic. Conversely: if $X$ has a $BP(n)_*$ resolution, $BP(n)_*(f) \circ \rho_A = f_2 \circ \rho_A$ will be epic forcing $\rho_X = \rho(n, \infty)$ to be epic. Thus hom dim_{BP_*}BP_*(X) $\leq n + 1$ (3.13).

§4. **Estimates of hom dim_{BP_*}BP_*(X)**

We collect some diverse ways of computing the invariant, hom dim_{BP_*}BP_*(X). The dimension estimate (4.4) which gives an upper bound of the invariant is new; the other estimates, both of which give lower bounds, are improvements or adaptations of techniques found in the papers of Conner and Smith. The section ends with a discussion of relations between the two lower bound estimating techniques and computation of examples.

**Proposition 4.1.** Let $X$ be a finite complex with $H^k(X;Z_{(p)})$ $T_0$ torsion free for $k \geq 2(p^a + \cdots + p + 1)$, then $\rho(n-1, n); BP(n)_*(X) \to BP(n-1)_*(X)$ is epic.
The usual spectral sequence,
\[ E_2^{ *, *}(X) \Rightarrow H^*(X; BP^*) \Rightarrow BP^*(X) \]
shows that \( BP\langle n \rangle^d(X) \) is \( T_0 \) torsion free for \( k \geq 2(p^n + \cdots + p + 1) \). Now let us consider the cohomology version of the \( BP \) Bockstein sequence

\[ \begin{array}{ccc}
BP\langle n \rangle^d(X) & \xrightarrow{\rho(n-1, n)} & BP\langle n-1 \rangle^d(X) \\
& \xrightarrow{\Delta_n} & BP\langle n \rangle^{d+2p^n-1}(X).
\end{array} \]  

(4.2)

\( \Delta_n \) is \( T_0 \) torsion valued (3.2); so \( \rho(n-1, n) \) is epic for \( s \geq 2(p^n + \cdots + p + 1) - (2p^n - 1) \). By the splitting theorem (2.2), \( \rho(n-1, n) \) is epic for \( s \leq 2(p^n-1 + \cdots + p + 1) \).

**Proposition 4.3.** Let a finite complex \( X \subseteq R^t \) have \( H_s(X; Z_{(p)}) \) \( T_0 \) torsion free for \( s < k \). If \( t < k + 2(p^n + \cdots + p + 1) \), then \( \text{hdim}_{BP_*} BP_*(X) \leq n \).

**Proof.** Let \( N \) be a regular neighborhood of \( X \subseteq R^t \). \( X \subseteq N \) is a homotopy equivalence and \( N \) is a \( t \) dimensional manifold with boundary. The tangent bundle of \( N \) is trivial; so \( N \) is \( BP \) orientable. Using the \( BP \) module structure of \( BP\langle m \rangle \), there are duality isomorphisms \( BP\langle m \rangle^d(N/\partial N) \cong BP\langle m \rangle^d(X^+). \) For \( q \geq 2(p^n + \cdots + p + 1) \), (implying \( t - q \leq k \)), the \( BP\langle 0 \rangle \) duality and the hypothesis imply \( H^d(N/\partial N; Z_{(p)}) \) is \( T_0 \) torsion free. Proposition 4.1 applies to \( N/\partial N \) to imply that \( \rho(n-1, n): BP\langle n \rangle^*(N/\partial N) \rightarrow BP\langle n-1 \rangle^*(N/\partial N) \) is epic. By (3.13), \( \text{hdim}_{BP_*} BP_*(X) = \text{hdim}_{BP_*} BP_*(X^+) \leq n \).

**Corollary 4.4** (Dimension estimate). If \( X \) is a \( q \) dimensional finite complex and if \( q < p^n + \cdots + p + 1 \), then \( \text{hdim}_{BP_*} BP_*(X) \leq n \).

**Proof.** This follows by letting \( k = 0 \) and \( t = 2q + 1 \) in Proposition 4.3. Note that \( q < p^n + \cdots p + 1 \) implies \( 2q + 1 \leq 2(p^n + \cdots + p + 1) \) as required.

**Corollary 4.5** (Adams, [1]; Conner and Smith [13]). If \( X \) is a finite complex, then \( \text{hdim}_{MU_*} MU_*(X) \) is finite.

**Proposition 4.6** (Ideal annihilator estimate). Let \( X \) be a connected CW spectrum with \( H_*(X; Z_{(p)}) \) of finite type. If \( x \cdot y = 0 \) for \( 0 \neq y \in BP_*(X) \), then \( \text{hdim}_{BP_*} BP_*(X) \geq n + 1 \).

**Proof.** Immediate from (3.5) and (3.13).

Let \( c: \mathcal{A}(p) \rightarrow \mathcal{A}(p) \) be the canonical antiautomorphism of the mod \( p \) Steenrod algebra. Since the Bockstein \( Q_0 \) (\( Q_0 = Sq^1 \) if \( p = 2 \)) is primitive, \( c(Q_0) = -Q_0 \). Thus \( c \) restricts to an antiautomorphism of \( (Q_0) \), the two-sided ideal generated by \( Q_0 \). The universal coefficient theorem isomorphism \( H^d(X; Z_p) \rightarrow \text{Hom}_{Z_p}(H_q(X; Z_p); Z_p) \) defines a dual pairing \( \langle \cdot, \cdot \rangle: H^q(X; Z_p) \otimes H_q(X; Z_p) \rightarrow Z_p \). For \( \alpha \in \mathcal{A}(p) \) of dimension \( r \) and \( x \in H_q(X; Z_p) \), define \( \alpha x \in H_{q-r}(X; Z_p) \) to be the homology class defined by (4.7).

\[ \langle y, \alpha x \rangle = \langle c(\alpha)y, x \rangle \text{ for all } y \in H^q(X; Z_p). \]  

(4.7)

This defines a left \( \mathcal{A}(p) \) module structure on \( H_*^q(X; Z_p) \).

**Theorem 4.8** (Steenrod operation estimate). Let \( X \) be a connected CW spectrum with \( H_*(X; Z_{(p)}) \) of finite type. If there is a selection of operations \( b_1, b_2, \ldots, b_k \) in \( (Q_0) \) such that the composition \( b_1b_2\ldots b_k \) acts nontrivially on the image of \( \rho(-1, n): BP\langle n \rangle_*(X) \rightarrow H_*^d(X; Z_p) \).
$H_*(X; Z_p)$, then $\text{hom dim}_{BP_*} BP_* (X) \geq n + k + 1$. In particular, if $b \rho(-1, n)(x) \neq 0$ for $b \in (Q_0)$ and $x \in BP_*(n)_*(X)$, then $x$ is not in the image of $\rho(n, \infty): BP_*(X) \to BP_*(n)_*(X)$.

**Proof.** A $BP$ resolution of $X, W \to A \xrightarrow{f} X$ induces the exact rows of the commutative diagram (4.9).

\[
\begin{array}{cccc}
BP_*(A) & \xrightarrow{f_1} & BP_*(X) & \to 0 \\
\downarrow \rho_1 & & \downarrow \rho_2 & \\
BP_*(n)_*(A) & \xrightarrow{f_2} & BP_*(n)_*(X) & \xrightarrow{\delta_3} BP_*(n)_*(W) \\
\downarrow \rho_3 & & \downarrow \rho_4 & \\
H_*(A; Z_p) & \xrightarrow{f_3} & H_*(X; Z_p) & \xrightarrow{\delta_3} H_*(W; Z_p)
\end{array}
\]  

Since $H_*(A; Z)$ is free abelian (3.11), $\rho_1$ and $\rho_3$ are epic (3.6 and 3.7) and $(Q_0)$ acts trivially on $H_*(A; Z_p)$. By hypothesis, $0 \neq b_1 b_2 \cdots b_k \rho_4(x)$ for some $x \in BP_*(n)_*(X)$.

**Claim 4.10.** $0 \neq \partial_3 b_2 \cdots b_k \rho_4(x) = b_2 \cdots b_k \partial_3 \rho_4(x) = b_2 b_k \rho_5 \rho_3(x)$.

**Proof of claim.** Suppose $\partial_3 b_2 \cdots b_k \rho_4(x) = 0$, then $b_2 \cdots b_k \rho_2(x) = f_3 \rho_3 \rho_1(a)$ for some $a \in BP_*(A)$. So $0 \neq b_1 b_2 \cdots b_k \rho_4(x) = b_1 f_3 \rho_3 \rho_1(a) = f_3 b_1 \rho_3 \rho_1(a)$ which gives the absurd conclusion that $b_1 \in (Q_0)$ acts non trivially on $H_*(A; Z_p)$.

Conclusion of proof. Thus $x \notin \text{Image } \rho_2 = \text{Image } \rho_2 \circ f_1 = \text{Image } f_2 \circ \rho_1$, else $\partial_2(x) = 0$ contradicting (4.10). If $k = 1$, the failure of $\rho_2$ to be epic implies $\text{hom dim}_{BP_*} BP_* (X) \geq n + 2$ (3.13). If $k > 1$, by an induction on the length of the compositions of operations of $(Q_0)$, $0 \neq b_2 \cdots b_k \rho_3 \partial_2(x)$ implies $\text{hom dim}_{BP_*} BP_* (W) \geq n + (k - 1) + 1$. This implies $\text{hom dim}_{BP_*} BP_* (X) \geq n + k + 1$.

By Spanier–Whitehead duality, we can do the entire homological dimension theory for Brown–Peterson cohomology modules over the cohomology coefficient ring $BP_*$. In general, the resulting invariant $\text{hom dim}_{BP_*} BP_* (X)$ is not equal to $\text{hom dim}_{BP_*} BP_* (X)$ (see (4.29).)

**Corollary 4.11 (Conner and Smith [11]).** Let $X$ be a finite complex. If there is a selection of operations $b_1, b_2, \ldots, b_k$ in $(Q_0)$ such that the composition $b_1 b_2 \cdots b_k$ acts non trivially in $H^*(X; Z_p)$, then both $\text{hom dim}_{BP_*} BP_* (X) \geq k$ and $\text{hom dim}_{BP_*} BP_* (X) \geq k$.

**Proof.** $b_1 b_2 \cdots b_k$ acting non trivially in $H^*(X; Z_p)$ implies $c(b_1) \cdots c(b_k) c(b_1)$ acts non trivially in $H_*(X; Z_p)$, Apply (4.8) and its dual.

We now have two techniques for computing lower bounds of $\text{hom dim}_{BP_*} BP_* (X)$. Neither is generally effective. One only has to resolve a complex $X$ having $\text{hom dim}_{BP_*} BP_* (X) = n + 1$ with a $BP_*$ resolution $W \to A \to X$ to obtain a complex $W$ with no $BP_*(W)$ annihilators and $\text{hom dim}_{BP_*} BP_* (W) = n$. Conner and Smith show how a stable homotopy class $\gamma \in \pi^3_{2m-1}$ of odd order $q$ which is in the image of the $J$ homomorphism gives a stable complex $X(\gamma) = S^0 \cup \beta^1 \cup e^3_1$ with $\text{hom dim}_{MU_*} MU_* (X(\gamma)) = 2[11; 12, Section 2; 14, 7.2]$. If $\gamma$ is not detected by a mod $p$ Hopf invariant, the Steenrod operation estimate will give the correct, but useless, information that $\text{hom dim}_{BP_*} BP_* (X(\gamma)) \geq 1$. 

How are the ideal annihilator estimate and the Steenrod operation estimate related? In general, not very well. In the special case when $0 = x_n y$ for $y \in BP_\ast(X)$ and $0 \neq \rho(-1, \infty)(y) \in H_\ast(X; \mathbb{Z}_p)$, a relation becomes clear however. The reader has probably recognized that the BP Bockstein sequence (2.1) has a "first differential"

$$BP(n-1)_\ast(X) \xrightarrow{\Delta_n} BP(n-1)_\ast s_{2p^n+1}(X) \xrightarrow{\rho(n-1,n)} BP(n-1)_\ast s_{2p^n+1}(X)$$

which gives an operation $\rho(n-1,n) \circ \Delta_n \in BP(n-1)_\ast s_{2p^n-1}(BP(n-1))$. A relation between the two estimates in special cases derives from the fact that this operation "covers" a non-zero class in $H^{2p^n-1}(BP(n-1); \mathbb{Z}_p) \cong Z_p$. In the following, we assume a familiarity with the notation of [23].

**Lemma 4.12.** Let $1 \in H^0(BP(n-1); Z_p) \cong Z_p$ be a generator, then a generator of $H^{2p^n-1}(BP(n-1); Z_p) \cong Z_p$ is $Q_0 = -c(Q_{p^n-1})I$.

**Proof.** From (2.5), it follows that $Q_0$ is trivial on $H^i(BP(n-1); Z_p)$ for $i < 2p^n - 2$. For $2 \leq m \leq n$ and $x \in H^{2m-2p^n}(BP(n-1); Z_p)$, $Q_0 \circ (p^n-1)x = 0$. Now we compute inductively that $Q_m x = -c(Q_{p^n-1}) x = -c(p^n-1) Q_{p^n-1} x = -c(Q_{p^n-1}) x + c(p^n-1)c(Q_{p^n-1}) x = 0$ (induction).

**Lemma 4.13.** $\rho(-1,0) : H^{2p^n-1}(BP(n-1); Z_p) \to H^{2p^n-1}(BP(n-1); Z_p)$ is an isomorphism.

**Proof.** It is easy to check that $H^{2p^n-1}(BP(n-1); Z_p)$ is a finite cyclic group ($a Z_p$) mapping onto the mod $p$ reduction. It suffices to show that $H^{2p^n-1}(BP(n-1); Z_p) \cong Z_p$ and this is done by using the classical Bockstein sequence (4.14) which defines $Q_0$.

$$H^{2p^n-2}(BP(n-1); Z_p) \xrightarrow{Q_0} H^{2p^n-1}(BP(n-1); Z_p) \to H^{2p^n-1}(BP(n-1); Z_p) \cong Z_p$$

(4.14)

By Lemma 4.12, $Q_0$ is an epimorphism. By exactness, $\rho$ is then monic as required.

**Diagram 4.15**

**Lemma 4.16** Diagram 4.15 commutes up to multiplication by a unit of $Z(p)$.

**Proof.** Both $\rho(-1,0) \circ \rho(0,n) \circ \Delta_n = \rho(-1,n) \circ \Delta_n$ and $c(Q_0) \circ c(p^n-1) \circ c(Q_{p^n-1})$ determine non-zero classes in $H^{2p^n-1}(BP(n-1); Z_p) \cong Z_p$ by (2.5) and (4.12), respectively. The triangle commutes up to sign. By Lemma 4.13, $\Delta_n \circ c(p^n-1) \circ c(Q_{p^n-1}) \circ \rho(-1,n-1)$ and $\rho(0,n) \circ \Delta_n$ both give non-zero classes in $H^{2p^n-1}(BP(n-1); Z_p) \cong Z_p$. 

COROLLARY 4.17. If \( \lambda a = \rho(-1, n)\Delta_n \cdots \Delta_1 \Delta_0(b) \neq 0 \) for some \( \lambda \in \mathbb{Z}_p \), then \( \lambda a = \rho(-1, n)\Delta_n \cdots \Delta_1 \Delta_0(b) \) for some \( \lambda \in \mathbb{Z}_p \).

Proof. By hypothesis, \( a = \rho(-1, n)\Delta_n \cdots \Delta_1 \Delta_0(b) = \rho(-1, n)\Delta_n \cdots \Delta_1 \Delta_0(b) \) using the fact from (4.16) that \( c(\mathcal{P}^r \cdots \mathcal{P}^1 Q_0) \circ \rho(-1, q) = \lambda' \rho(-1, q + 1) \circ \Delta_{q+1} \) where \( \lambda' \) and \( \lambda'' \) are non-zero elements of \( \mathbb{Z}_p \).

Example 4.18. Let \( t_\ast \in H^n(K(Z_\ast, n); Z \_p) \) be the fundamental class and let \( f: B(Z_\ast)^n = K(Z_\ast, 1) \times \cdots \times K(Z_\ast, n) \to K(Z_\ast, n) \) classify the \( n \)-fold external product \( t_1 \times \cdots \times t_1 \in H^n(B(Z_\ast)^n; Z \_p) \). For \( 0 \leq q \leq n - 2 \), \( 0 \neq Q_0(\mathcal{P}^1 Q_0) \cdots (\mathcal{P}^r \cdots \mathcal{P}^1 Q_0)(t_1 \times \cdots \times t_1) = f^*(Q_0(\mathcal{P}^1 Q_0) \cdots (\mathcal{P}^r \cdots \mathcal{P}^1 Q_0)(t_1) \circ \cdots \circ t_1) \). By (4.17), \( \Delta_{q+1}: BP(q)_{n+1-p+1} \to BP(q+1)_{n}(K(Z_\ast, n)) \) is non-zero. Let \( \sigma_n \in BP_n(K(Z_\ast, n)) \equiv Z \_p \) be a generator. We see that \( x_{q+1} \circ (q + 1, n) = 0 \) for \( 0 \leq q \leq n + 2 \). For dimensional reasons, this implies that \( x_{n+1} = 0 \). So \( p, x_1, \ldots, x_{n-1} \) all annihilate \( \sigma_n \). This fact is essentially Theorem A of [29].

We shall use the following technical proposition for computing examples. It is a special case of Corollary 2.4 of [14] designed to deal with the important class of stable complexes of form \( \mathcal{S}^0 \cup e^1 \cup e^{2n+1} \cup \cdots \cup e^{2n+1} \).

PROPOSITION 4.19. Let \( Y \) be a connected CW spectrum with \( Z(p) \) homology of finite type and satisfying the following.

(i) \( H_i(Y; Z(p)) = 0 \) for \( i < k \) or \( i = k + 1 \).

(ii) \( H_k(Y; Z(p)) \) is free for \( s > 0 \), with generator \( \sigma_0 \).

(iii) \( H_i(Y; Z(p)) \) is \( Z(p) \)-free for \( i > k + 1 \).

Let \( \sigma_n = \rho(0, n)\Delta_n^{-1}(\sigma_0) \in BP(n)_{k}(Y) \). Let \( \tau \in H^{k+1}(Y; Z_\ast) \cong Ext(H_k(Y; Z(p)); Z \_p) \) be a generator. We conclude: \( x_n \circ \tau = 0 \) if and only if \( \mathcal{P}^{r-1} \cdots \mathcal{P}^1 \tau \neq 0 \).

Proof. By the exact sequence (4.20), \( \tau = \Delta_n^{-1}(\tau \circ \sigma_0) \in H_{k+1}(Y; Z \_p) \) is a generator.

\[
0 = H_{k+1}(Y; Z(p)) \xrightarrow{\Delta_n^{-1}} H_k(Y; Z(p)) \xrightarrow{\tau} H_k(Y; Z(p)).
\]

Without loss of generality, we may choose \( \tau \) to be dual to \( \tau \). In what follows, we shall ignore "up to multiplication by a unit of \( Z(p) \)." By (4.16), the diagram (4.21) commutes; the sequence is exact.

\[
\begin{array}{cccc}
BP(q-1)_{k+2p-1}(Y) & \xrightarrow{\Delta_n^{-1}} & BP(q)_k(Y) & \xrightarrow{\rho(0, q)} & BP(q)_{k+2p-2}(Y) \\
\| & & \| & & \| \\
\rho(-1, q-1) & H_k(Y; Z(p)) & H_k(Y; Z(p)) & \Delta_n & H_k(Y; Z(p)) \\
\| & \downarrow & \| & \downarrow & \| \\
H_{k+2p-1}(Y; Z(p)) & c(\mathcal{P}^{r-1} \cdots \mathcal{P}^1 \mathcal{P}^0) & H_{k+1}(Y; Z(p)) & \end{array}
\]
If \( x_n p^{n-1} \sigma_n = 0 \), then \( p^{n-1} \sigma_n = \Delta_n(y) \) for some \( y \in BP(n-1)_{k+2p^{n-1}}(Y) \). So \( p^{n-1} \sigma_0 = 0 \) and \( p(0, n)(p^{n-1} \sigma_n) = \rho(0, n)\Delta_n(y) = \Delta_0 c(\rho p^{n-1} \cdots \rho p^{1})\rho(-1, n-1)(y) \) and so \( \rho p^{n-1} \cdots \rho p \epsilon \neq 0 \).

We now assume \( \rho p^{n-1} \cdots \rho p^{q-1} \neq 0 \). We prove the converse by proving inductively (4.22).

4.22. If \( 0 \neq y \in BP(2)_{q}(Y) \) and \( x_q \cdot y = 0 \), then \( y = \lambda p^{q-1} \sigma_q = \epsilon(0, q) \rho q \), \( \lambda = \epsilon(0, q) \rho q \), \( \epsilon(0, q) \rho q \neq 0 \). We prove the statement for \( 0, 1, \ldots, q-1 < n \). Analysis of the \( BP \) Bockstein sequences (2.1) in the light of this assumption shows that in dimension \( k + 2p^{n-1} - 1 \), \( \rho(-1, 0) \circ \rho(0, q-1) \circ \cdots \circ \rho(q-2, q-1) \) is epic. Thus both paths of (4.21) are non-zero, so \( \epsilon = \epsilon(0, q) \rho q \neq 0 \) is a unit of \( \Z_{(p)} \). So \( u = 1 \) and \( \rho(q-1, 0) \circ \cdots \circ \rho(q-2, 0) \circ \cdots \circ \rho(q-2, q-1) \) is non-zero in dimension \( k \), Image \( \epsilon = \epsilon(0, q) \rho q \) is in the image of \( \epsilon \). Thus \( 0 = x_q z \) and \( t = 0 \). We conclude \( y = \epsilon p^{q-1} \sigma_q \) as required to confirm (4.22) for \( q \).

Example 4.23. Using the first two elements of Hopf invariant one, Conner and Smith construct a stable complex \( X(n, v) = S^0 \cup_{2} e^1 \cup_{4} e^3 \cup_{5} e^7 \) [11, p. 480; 12, Section 3; 14, pp. 166–168]. The homology of \( X(n, v) \) satisfies the hypothesis of (4.19). Conner and Smith show that 

\[
\rho_{q} = q^{4}S^{1}S^{2}S^{4}: H^{0}(X(n, v); Z_{2}) \rightarrow H^{7}(X(n, v); Z_{2})
\]

is an isomorphism. Recalling \( \rho_{2} = \rho \) and \( \rho_{1} = \rho \), (4.19) says that \( x_1 \sigma_1 = 0 \) and \( x_2 \cdot \sigma_2 = 0 \) where \( \sigma_1 \in BP(1)_{0}(X(n, v)) \cong Z_{2} \) and \( \sigma_2 \in BP(2)_{0}(X(n, v)) \cong Z_{2} \) are the nonzero elements. By \( BP \) Bockstein sequences (2.1), these two facts imply \( BP(1)_{2}(X(n, v)) = 0 \) and \( \rho(1, 2): BP(2)_{2}(X(n, v)) \rightarrow BP(1)_{2}(X(n, v)) \) is not epic. Thus hom \( \dim_{BP} BP_{*}(X(n, v)) \geq 3 \) (3.13). (Note that \( c(S^{4}S^{2}S^{4}) : \rho(1, 1, 1) \neq 0 \); so we have actually used the Steenrod operation estimate (4.8).) To be gross, \( S^{15}X(n, v) \) is an “honest” 22-dimensional complex. Apply (4.3) with \( t = 45 \) and \( k = 15 \). \( 45 \leq 15 + 2(2^3 + 2^2 + 2 + 1) \) implies hom \( \dim_{BP} BP_{*}(S^{15}X(n, v)) \) is the best possible.

Remark 4.24. The only torsion of \( H_{*}(X(n, v); Z) \) is 2-primary. For \( p = 2 \), \( BP(1)_{*} = k_{*}(X(n, v)) \otimes Z_{(2)} \). Our analysis in (4.23) gives that \( k_{*}(X(n, v)) \otimes Z_{(2)} = 0 \) and that \( \zeta: MU_{*}(X(n, v)) \rightarrow k_{*}(X(n, v)) \) fails to be an epimorphism. In Baas’ work on bordism with singularities, he constructed a tower of spectra

\[
MU \rightarrow \cdots \rightarrow MU_{*}(2) \rightarrow MU_{*}(1) \rightarrow MU_{*}(0) = KZ
\]

and asked whether analogs of the Conner–Smith theorems held [6]. The proof of (2.7) implies \( MU_{*}(1)_{*} = k_{*}(X(n, v)) \otimes Z_{(2)} \) and \( k_{*}(X(n, v)) \otimes Z_{(2)} \) are isomorphic; so \( MU_{*}(1)_{*}(X(n, v)) = 0 \) and \( \zeta: MU_{*}(X(n, v)) \rightarrow MU_{*}(1)_{*}(X(n, v)) \) is not epic. The rows and columns in (4.26) are \( MU \)-Bockstein sequences and \( X = X(n, v) \).
\[ \begin{align*}
\text{MU}_1(X) & \quad \text{MU}_2<1>_{-2}(X) = 0 \\
\mu & \text{ epic and } \zeta \text{ is not; so } \psi: \text{MU}_1(X) \to \text{MU}_2<1>_{-2}(X) \text{ fails to be epic even though hom dim}_{\text{MU}_*} \text{MU}_*(X(\eta, \nu)) = 3. \text{ This shows that the Conner-Smith program cannot be generalized by a } \text{MU} \text{ tower of spectra without localization.}
\end{align*} \]

**Example 4.27.** Fix a prime \( p \). There is some least integer \( s \) such that a stable map \( f'': \text{CP}(1) = S^2 \to S^2 \) of degree \( p'q, \gcd(p, q) = 1 \), extends to a stable map \( f: \text{CP}(p^n) \to S^2 \). Let \( Y \) be the cofibre of \( f \). \( H_*(Y; Z_{(p)}) = 0 \) for \( i < 2 \) and \( i = 3 \). \( H_2(Y; Z_{(p)}) \approx Z_p \). For \( i > 3, H_i(Y; Z_{(p)}) \) is \( Z_{(p)} \) free. \( \delta: H^i(\text{CP}(p^n); Z_p) \to H^{i+1}(Y; Z_p) \) is an isomorphism for \( i \geq 2 \). Let \( t \in H^2(\text{CP}(p^n); Z_p) \) be a generator, then \( t^{p^n} \neq 0 \). \( \mathcal{P}_0 \mathcal{P}_1 \cdots \mathcal{P}_i \delta(t) = \delta(\mathcal{P}_0 \mathcal{P}_1 \cdots \mathcal{P}_i t) = \cdots = \delta(\mathcal{P}_0 \mathcal{P}_1 \cdots \mathcal{P}_i t^{p^n}) = \delta(t^{p^n}) \neq 0 \). By (4.19) and (3.13), hom dim \( \text{BP}_*, \text{BP}_*(Y) \geq n + 1 \). The stable dimension of \( Y \) is \( 2p^n - 1 \); so it can be represented as an "honest" complex of dimension \( 4p^n - 2, 4p^n - 2 < p^{n+1} + \cdots + p + 1 \) for positive \( n \); thus our dimension estimate (4.4) assures us the above lower bound is the best possible and hom dim \( \text{BP}_*, \text{BP}_*(Y) = n + 1 \). This shows that the upper bound estimate of Corollary 4.4 is, in fact, a good one.

**Remark 4.28.** This computation of hom dim \( \text{BP}_*, \text{BP}_*(Y) \) is the first example known to the authors of a complex with a high hom dim \( \text{BP}_*, \text{BP}_*(Y) \) which is known precisely.

**Remark 4.29.** \( H^*(Y; Z) \) is concentrated in odd dimensions; so the usual spectral sequence

\[ E^2_{\ast \ast}(Y) = H^*(Y; \text{MU}^*) \Rightarrow \text{MU}^*(Y) \]

collapses and \( \text{MU}_*(Y) \to H^*(Y; Z) \) is epic. This hom dim \( \text{MU}_*, \text{MU}_*(Y) = 1 \) while hom dim \( \text{MU}_*, \text{MU}_*(Y) \) is high.

**Remark 4.30.** The lower bound of hom dim \( \text{BP}_*, \text{BP}_*(Y) \) could be done using the annihilator ideal data found on p. 173 of [14]. One needs only to localize the data at the prime \( p \) and then recall the Milnor manifolds \( V^{2p^n-2} \) represent polynomial generators of \( \text{MU}_* \otimes Z_{(p)} \).

### §5. ASSORTED PRODUCTS AND SPECTRAL SEQUENCES

The aims of this section are twofold. First, we shall complete the proof of Theorem 1.1 and then we shall construct a variety of spectral sequences.

**Lemma 5.1.** Let \( Y \) be a connected CW spectrum with \( H_*(Y; Z_{(p)}) \) a free \( Z_{(p)} \) module of finite type, then for any CW spectrum \( X \)

\[ \rho(\widetilde{\eta}, \infty): \text{BP}_*(Y) \otimes \text{BP}_* \text{BP}<\eta>_{\ast}(X) \to \text{BP}<\eta>_{\ast}(Y \wedge X) \]
and

\[ \rho(n, \infty): BP<\langle n \rangle \rangle(X) \otimes_{BP_*} BP_* (Y) \to BP<\langle n \rangle \rangle(X \wedge Y) \]

are isomorphisms.

Proof. \(BP_* (Y)\) is a free \(BP_* \) module (3.10); so both \(BP_* (Y) \otimes_{BP_*} BP<\langle n \rangle \rangle(-)\) and \(BP<\langle n \rangle \rangle(Y \wedge -)\) are homology theories. When \(X = S^0\), \(\rho(n, \infty)\) is proved to be an isomorphism by bookkeeping arguments. By the uniqueness theorem for homology theories, \(\rho(n, \infty)\) is an isomorphism for all \(X\). The proof for \(\rho(n, \infty)\) is the same.

Lemma 5.2. Let (5.3) be a commutative diagram with exact rows. If \(\alpha\) and \(\beta\) are isomorphisms, then

\[
\begin{array}{c}
0 \to C_1 \xrightarrow{\gamma} C_2 \xrightarrow{h^{-1} \alpha^{-1} \beta} T \to 0
\end{array}
\]

is exact.

Proof: Routine.

Proposition 5.4. Let \(X\) and \(Y\) be connected CW spectra with \(\mathbb{Z}_{(p)}\) homology of finite type. If either (i) \(\text{hom dim}_{BP_*} BP_* (X) \leq n + 2\) and \(\text{hom dim}_{BP_*} BP_* (Y) = 0\) or (ii) \(\text{hom dim}_{BP_*} BP_* (X) \leq 1\), hold, then there is a natural short exact sequence (5.5).

\[
0 \to BP_* (X) \otimes_{BP_*} BP<\langle n \rangle \rangle (Y) \xrightarrow{\rho(n, \infty)} BP<\langle n \rangle \rangle(X \wedge Y) \to \text{Tor}_{1, BP_*} (BP_* (X), BP<\langle n \rangle \rangle (Y)) \to 0.
\]

(5.5)

If \(\text{hom dim}_{BP_*} BP_* (X) \leq n + 1\) and \(\text{hom dim}_{BP_*} BP_* (Y) = 0\), then \(\rho(n, \infty)\) is an isomorphism.

Proof. First, we choose a \(BP_* \) resolution of \(X\), giving us a short exact sequence

\[
0 \to BP_* (W) \xrightarrow{g_*} BP_* (A) \xrightarrow{f_*} BP_* (X) \to 0
\]

with \(BP_* (A)\) a free \(BP_* \) module. This induces the commutative diagram (5.6).

\[
0 \to T \to BP_* (W) \otimes_{BP_*} BP<\langle n \rangle \rangle (Y) \xrightarrow{g_*} BP_* (X) \otimes_{BP_*} BP<\langle n \rangle \rangle (Y) \to 0
\]

(5.6)

\[
T = \text{Tor}_{1, BP_*} (BP_* (X), BP<\langle n \rangle \rangle (Y)).
\]
By (5.1), $\rho_A$ is an isomorphism. If (ii) holds, then $BP_*(W)$ is also $BP_*$ free and so $\rho_W$ is an isomorphism. So (ii) and Lemma 5.2 imply the exactness of (5.5). If $\hom dim_{BP_*}BP_*(Y) = 0$ and if $0 < \hom dim_{BP_*}BP_*(X) \leq n + 1$, $f_2 = BP_*(f) \otimes_{BP_*}BP_*(Y)$ is epic by (5.1) and (3.15) So $\partial = 0$ and $\rho$ is an isomorphism by induction on $\hom dim_{BP_*}BP_*(\ )$ implying $\rho_X$ is an isomorphism by the “five” lemma. So if (i) holds, $\rho_w$ and $\rho_A$ are isomorphisms and the exactness of (5.5) follows again from Lemma 5.2.

**Proposition 5.7.** Let $X$ be a connected CW spectrum with $Z(p)$ homology of finite type. There is a natural short exact sequence (5.8)

$$0 \to BP_n(X) \otimes_{Z(p)[x_{n+1}]} Z(p) \to BP_n(X) \to \text{Tor}^{Z(p)[x_{n+1}]} \to BP_n(X) \to Z(p) \to 0 \quad (5.8)$$

$\rho(n, n+1)$ is an isomorphism if and only if $\hom dim_{BP_*}BP_*(X) \leq n + 1$.

**Proof.** Apply the functor $BP_*(X) \otimes_{Z(p)[x_{n+1}]} Z(p)$ to the $Z(p)[x_{n+1}]$ free resolution of $Z(p)$,

$$0 \to Z(p)[x_{n+1}] \to BP_*(X) \to Z(p) \to 0$$

to yield:

$$T = \text{Tor}^{Z(p)[x_{n+1}]}(BP_n(X); Z(p))$$

$$0 \to T \to BP_*(X) \to BP_*(X) \to BP_*(X) \otimes_{Z(p)[x_{n+1}]} Z(p) \to 0$$

The exact sequence (5.8) follows from (5.2). Note that $\rho(n, n+1)$ is epic if and only if $\rho(n, n+1)$ is epic which occurs if and only if $\hom dim_{BP_*}BP_*(X) \leq n + 1$ by (3.8; 3.13).

**Proposition 5.10.** Let $X$ be a finite complex. If $\text{Tor}^{BP_*(X)}BP_n(X), BP_*(X) = 0$, then $\hom dim_{BP_*}BP_*(X) \leq n + 1$.

**Proof.** The exact sequence

$$0 \to BP_*(X) \to BP_*(X) \to BP_*(X) \to 0$$

induces the exact sequence

$$\text{Tor}^{BP_*}_{m+2}(BP_*(X), BP_*(X)) \to \text{Tor}^{BP_*}_{m+2}(BP_*(X), BP_*(X)) \to \text{Tor}^{BP_*}_{m+2}(BP_*(X), BP_*(X)).$$

By an induction on dimensions, $\text{Tor}^{BP_*}_{m+2}(BP_*(X), BP_*(X)) = 0$ implies $\text{Tor}^{BP_*}_{m+2}(BP_*(X), BP_*(X)) = 0$. By iteration, this implies $\text{Tor}^{BP_*}_{m+2}(BP_*(X), BP_*(X)) = 0$. 
for \( m \geq n \). hom \( \text{dim}_{BP}BP_\bullet(X) \) is a finite number (4.5). Suppose \( \text{hom dim}_{BP}BP_\bullet(X) = m + 1 > n + 1 \). The short exact sequence

\[
0 \to BP\langle m \rangle_\bullet \xrightarrow{\cdot x_m} BP\langle m \rangle_\bullet \to BP\langle m - 1 \rangle_\bullet \to 0
\]

induces the top exact row in the commutative diagram:

\[
\begin{array}{ccc}
0 &=& 0 \\
\uparrow & & \uparrow
\end{array}
\begin{array}{ccc}
BP\langle m \rangle_\bullet (X) &=& \text{Tor}_1^{BP}(BP_\bullet(X), BP\langle m - 1 \rangle_\bullet) \\
\downarrow \rho(m, \infty) & & \downarrow \rho(m, \infty)
\end{array}
\begin{array}{ccc}
& & \\
BP\langle m \rangle_\bullet (X) & \xrightarrow{\cdot x_m} & BP\langle m \rangle_\bullet (X)
\end{array}
\]

(5.11)

The \( \rho(m, \infty) \)'s are isomorphisms (5.4); so multiplication \( x_m \) is monic in \( BP\langle m \rangle_\bullet (X) \). Thus hom \( \text{dim}_{BP}BP_\bullet(X) \leq m \) (3.13) which is a contradiction. So the supposition \( \text{hom dim}_{BP}BP_\bullet(X) > n + 1 \) is false.

**Proposition 5.12.** Let \( X \) be a finite complex. The following conditions are equivalent:

(i) \( \text{hom dim}_{BP}BP_\bullet(X) \leq n + 1 \);

(ii) \( \rho(n, \infty): BP_\bullet(X) \otimes_{BP} BP\langle n \rangle_\bullet \to BP\langle n \rangle_\bullet (X) \) is an isomorphism;

(iii) \( \text{Tor}_1^{BP}(BP_\bullet(X), BP\langle n \rangle_\bullet) = 0 \);

(iv) \( \text{Tor}_j^{BP}(BP_\bullet(X), BP\langle n \rangle_\bullet) = 0, j > 0 \);

(v) \( \rho(n, n + 1): BP\langle n + 1 \rangle_\bullet (X) \otimes \mathbb{Z}(\rho, \mathbb{Z}_\infty) \to BP\langle n \rangle_\bullet (X) \) is an isomorphism;

(vi) \( \text{Tor}_j^{BP}(BP_\bullet(W), BP\langle n + 1 \rangle_\bullet (X), Z(\rho)) = 0 \);

(vii) \( \text{Tor}_j^{BP}(BP_\bullet(W), BP\langle n + 1 \rangle_\bullet (X), Z(\rho)) = 0, j > 0 \).

**Proof.** The following implications have been established: (i) \( \Rightarrow \) (ii) (5.4); (i) \( \Rightarrow \) (iii) (5.4); (iii) \( \Rightarrow \) (i), (5.10); (iv) \( \Rightarrow \) (iii), (trivial); (i) \( \Leftrightarrow \) (v) \( \Leftrightarrow \) (vi), (5.7); and (vii) \( \Rightarrow \) (vi), (trivial). We now demonstrate (iii) \( \Rightarrow \) (iv) and (ii) \( \Rightarrow \) (i). The reader may complete the proof of (vi) \( \Rightarrow \) (vii) to finish the proposition.

To prove (ii) \( \Rightarrow \) (i), note that in diagram (5.6) with \( Y = S^0 \), \( \rho_X \) epic implies \( f_2 \) is epic and thus \( g_2 \) to be monic. Just as in the proof of (3.13), \( BP\langle n \rangle_\bullet (W) \) is \( T_\bullet \) torsion free and hom \( \text{dim}_{BP}BP_\bullet(X) \leq n + 1 \). If \( W \to A \to X \) is a \( BP_\bullet \) resolution of \( X \) and if \( j > 1 \), then there is an isomorphism

\[
\bar{\partial}: \text{Tor}_1^{BP}(BP_\bullet(X), BP\langle n \rangle_\bullet) \to \text{Tor}_1^{BP}(BP_\bullet(W), BP\langle n \rangle_\bullet).
\]

The implication (i) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) is proved by induction over hom \( \text{dim}_{BP}BP_\bullet(X) \).

**Remark 5.13.** For any CW spectrum \( X \), there is a natural isomorphism

\[
T_\bullet^{-1} \rho(n, \infty): BP_\bullet(X) \otimes_{BP} T_\bullet^{-1} BP\langle n \rangle_\bullet \to T_\bullet^{-1} BP\langle n \rangle_\bullet (X) \).
\]

**Proof.** If \( A \) is a finite complex with \( H_\bullet(A; Z(\rho)) \) \( Z(\rho) \) free, then \( T_\bullet^{-1} \rho(n, \infty) \) is an isomorphism by (5.1). If \( W \to A \to X \) is a \( BP_\bullet \) resolution for a finite complex \( X \), then it is also a \( T_\bullet^{-1} BP\langle n \rangle_\bullet \) resolution for \( X \) (use (3.7) and the proof of (3.15). Now one forms a diagram
like (5.6) and establishes the remark for finite complexes by induction on the homological dimension. It generalizes to arbitrary $CW$ spectra by (3.1).

Now we sketch the constructions of some spectral sequences related to our work. In this paper, our proofs do not employ these, but we have used such spectral sequences for first proofs and we feel these might be of independent interest in future investigations. For example, the original proofs of (3.6) and (6.5) used the following spectral sequence.

**Proposition 5.14.** Let $X$ be a connected $CW$ spectrum. There is a natural first quadrant spectral sequence

$$E^2_{i,j}(X) \cong BP^i \langle n - 1 \rangle \langle j \rangle \otimes (Z_p[x_n])_j \Rightarrow BP^i \langle n \rangle \langle j \rangle (X).$$

The differentials are $T_{i,j}$ torsion valued and the spectral sequence collapses if and only if $\rho(n - 1, n): BP^i \langle n \rangle \langle j \rangle (X) \rightarrow BP^i \langle n - 1 \rangle \langle j \rangle (X)$ is epic.

**Proof.** Apply Corollary 2.5 of [17] to the $BP$ Bockstein sequence (2.1). Each differential of the spectral sequence involves a $T_{n-1}$ torsion valued $\Delta_n$ (3.2) in its definition. (N.B. A key to sanity is to use the indexing conventions of the first derived exact couple of the exact couple on page 337 of [21].)

We shall now establish some spectral sequences which arise from the geometric resolution of a complex. The first such spectral sequence takes the form

$$E^2_{i,j}(X, Y) \cong \text{Tor}^{BP^i}_{i,j}(BP_*(X), BP^i \langle n \rangle \langle j \rangle (Y)) \Rightarrow BP^i \langle n \rangle \langle j \rangle (X \wedge Y)$$

for $-1 \leq n \leq \infty$. For $n = \infty$ or $Y = S^0$, this spectral sequence appears in both [1] and [13]. Our main contribution is to put in the module theories $BP^i \langle n \rangle \langle j \rangle (Y)$ in a more significant way. The proofs are essentially those found in [1] and [13]; so we offer only sketches. We shall assume $X$ and $Y$ are finite complexes, although Lemma 5 of [20] may be used to extend the results to a more general setting.

**Proposition 5.15.** There is a natural first quadrant spectral sequence

$$E^2_{i,j}(X, Y) = \text{Tor}^{BP^i}_{i,j}(BP_*(X), BP^i \langle n \rangle \langle j \rangle (Y)) \Rightarrow BP^i \langle n \rangle \langle j \rangle (X \wedge Y)$$

with edge homomorphism

$$BP_*(X) \otimes_{BP_0} BP^i \langle n \rangle \langle j \rangle (Y) = E^2_{i,j}(X, Y) \rightarrow E^2_{0,n}(X, Y) \rightarrow BP^i \langle n \rangle \langle j \rangle (X \wedge Y)$$

identified with the external product. $BP_*$ acts on $E^i_{j,*}(X, Y)$ commuting with differentials (which have bidegree $(-r, r - 1)$).

**Sketch proof.** We establish a diagram

$$
\begin{align*}
\ast & \leftarrow W_{m-1} \leftarrow \cdots \leftarrow W_0 \leftarrow W_{-1} = X \\
& \downarrow f_m \downarrow f_{m-1} \downarrow f_1 \downarrow f_0 \\
& A_m \leftarrow A_{m-1} \leftarrow \cdots \leftarrow A_1 \leftarrow A_0
\end{align*}
$$

where each cofibration

$$W_j \xrightarrow{q_j} A_j \xrightarrow{f_j} W_{j-1}$$

with

$$
\begin{align*}
q_j \circ s_j & = X, \\
f_j \circ s_j & = X
\end{align*}
$$

and the differentials $d_j$ are defined by

$$d_j : W_j \rightarrow W_{j-1}$$

for $j \geq 1$.
is a $BP_\ast$ resolution of $W_{j-1}$ in the sense of (3.11) ($f_{\ast j}$ is epic and $BP_\ast(A_j)$ is $BP_\ast$-free.)

$$0 \to BP_\ast(A_m) \overset{(m_{-1} \cdot f_m)}{\to} BP_\ast(A_m-1) \to \cdots \to BP_\ast(A_1) \overset{(m_0 \cdot f_1)}{\to} BP_\ast(A_0) \overset{f_0}{\to} BP_\ast(X) \to 0$$

is a free $BP_\ast$ resolution of the module $BP_\ast(X)$. Apply the functor $BP_\langle n \rangle_\ast(\ast \wedge \ast)$ to the diagram (5.16) to obtain an exact couple with $E_2^{j, k}(X, Y) = BP_\langle n \rangle_\ast(A_j \wedge Y) \cong BP_\ast(A_j) \otimes_{BP_\ast} BP_\langle n \rangle_\ast(Y)$ (5.1) and $d_{j} = (g_{j-1} \circ f_j) \otimes 1$. Identification of $E_2^{j, k}(X, Y)$ and convergence of the resulting spectral sequence are then routine. The proof of naturality is naturally tedious and is done in detail in [13].

**Proposition 5.17.** There is a natural spectral sequence $E_2^{j, k}(X, Y) = \text{Ext}_{BP_\ast}(BP_\ast(X), BP_\langle n \rangle_\ast(Y)) \Rightarrow BP_\langle n \rangle_\ast(X \wedge Y)$ with edge homomorphism $BP_\langle n \rangle_\ast(X \wedge Y) \to E_2^{j,*}(X, Y) \subset E_1^{j,*}(X, Y) = \text{Hom}_{BP_\ast}(BP_\ast(X), BP_\langle n \rangle_\ast(Y))$ induced by the slant product (see pp. 258f of [37]). $BP_\ast \cong BP^{-}\ast$ acts on $E_1^{j,*}(X, Y)$ and commutes with the differentials (which have bidegree $(r, 1-r)$).

**Proof.** The proof is analogous to that of (5.15) with the following lemma replacing (5.1).

**Lemma 5.18.** Let $A$ be a finite complex with $BP_\ast(A)$ free $BP_\ast$. For any finite complex $Y$ there is a natural isomorphism induced by the slant product $BP_\langle n \rangle_\ast(A \wedge Y) \cong \text{Hom}_{BP_\ast}(BP_\ast(A), BP_\langle n \rangle_\ast(Y))$.

**Proof.** First establish the isomorphism for $Y = S^0$ and then use the uniqueness theorem for cohomology theories.

**Remark 5.19.** There are dual spectral sequences to (5.15) and (5.17): just change the upper stars to lower stars and vice versa. They can be obtained by applying $S$-duality to (5.15) and (5.17) since we are using finite complexes; or, one can build a $BP^\ast$ geometric resolution for $BP^\ast(X)$ directly. Because $BP_\ast(X)$ and $BP^\ast(X)$ can have dramatically different module structure (see 4.29), it may be possible to play one against the other.

**Remark 5.20.** The spectral sequences obtained above can be done in much greater generality. Using Sullivan's theory of manifolds with singularities [7], we can kill off selected polynomial generators of $MU_\ast$ to obtain a homology theory $MUS_\ast(\cdot )$. $MUS$ is then a module theory over $MU$. The same proofs give spectral sequences once we replace $BP$ by $MU$ and $BP_\langle n \rangle$ by $MUS$.

§6. SPHERICAL CLASSES IN $BP_\ast(X)$

Choose generators $i_0 \in BP_m(S^m)$, compatible with suspensions, and let $i_n = \rho(n, \infty)i_\infty \in BP_\langle n \rangle_\ast(S^m)$. For each $-1 \leq n \leq \infty$, there is a natural Hurewicz homomorphism $\eta_n$: $\pi_\ast^\infty(S^n) \to BP_\langle n \rangle_\ast(S^n)$ which sends the class of a stable map $f: S^n \to X$ to $f_\ast i_n \in BP_\langle n \rangle_\ast(X)$. A class of $BP_\langle n \rangle_\ast(X)$ is said to be spherical if it is in the image of $\eta_n$.

**Theorem 6.1.** Let $X$ be a finite complex. The Hurewicz homomorphism

$$\eta_n: \pi_\ast^\infty(BP \wedge X) = BP_\ast(X) \to BP_\langle n \rangle_\ast(BP \wedge X)$$

is a monomorphism if hom dim$_{BP_\ast}BP_\ast(X) \leq n$. 
Proof. We do the \( n = \infty \) case first. \( \eta_\infty : \pi^*_\infty(BP \wedge X) \to \pi_\infty(BP \wedge BP \wedge X) = BP_\infty(BP \wedge X) \) has a left inverse induced by the pairing \( BP \wedge BP \to BP \). So \( \eta_\infty \) is split monic.

Now suppose \( f: S^n \to BP \wedge X \) is a stable map representing a class in the kernel of \( \eta_n \). \( BP_\infty(f)(1_\infty) = 0 \) implies \( BP_\infty(f) \equiv 0 \). Let \( \eta_n \infty : BP \wedge X \to C \to S^{n+1} \) be the stable cofibration sequence induced by \( f \). It then induces a short exact sequence:

\[
0 \to BP_\infty \langle n \rangle \langle BP \wedge X \rangle \xrightarrow{\cdot g_1} BP_\infty \langle n \rangle \langle C \rangle \to BP_\infty \langle n \rangle \langle (S^{n+1}) \rangle \to 0.
\]

Now we assume \( \hom \dim_{BP_\infty} BP_\infty(X) \leq n \). Since \( H_\infty(BP; Z_{(p)}) = Z_{(p)} \)-free, \( BP_\infty \langle BP \wedge X \rangle \cong BP_\infty(BP) \otimes_{BP_\infty} BP_\infty(BP \wedge X) \) (5.1). \( x_n \) acts monomorphically on \( BP_\infty(BP \wedge X) \) (3.13) and \( BP_\infty(BP) \otimes_{BP_\infty} \) is an exact functor; so \( x_n \) multiplication is monic in \( BP_\infty(BP \wedge X) \). It is certainly monic in \( BP_\infty(BP \wedge (S^{n+1})) \); so a simple application of the “five” lemma to two copies of (6.2) implies that \( x_n \) multiplication is monic in \( BP_\infty(BP \wedge C) \). Recall that if \( x_n \) multiplication is monic in \( BP_\infty(BP \wedge Y) \), then \( x_{n+1} \) multiplication is monic in \( BP_\infty(BP \wedge (Y + 1)) \) for \( Y = BP \wedge X \) or \( C \) (3.6). Thus the columns of commutative diagram (6.3) are exact.

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & 0 \\
\downarrow & & \downarrow \\
BP_\infty \langle n + 1 \rangle \langle BP \wedge X \rangle & \xrightarrow{\cdot g_3} & BP_\infty \langle n + 1 \rangle \langle C \rangle \\
\downarrow \cdot x_{n+1} & & \downarrow \cdot x_{n+1} \\
BP_\infty \langle n + 1 \rangle \langle BP \wedge X \rangle & \xrightarrow{\cdot g_2} & BP_\infty \langle n + 1 \rangle \langle C \rangle \\
\downarrow \rho(n, n+1) & & \downarrow \rho(n, n+1) \\
0 \to BP_\infty \langle n \rangle \langle BP \wedge X \rangle & \xrightarrow{\cdot g_1} & BP_\infty \langle n \rangle \langle C \rangle \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

By an induction on dimensions, beginning in negative dimensions, \( g_3 \) is monic. So \( g_2 \) is monic by the “five” lemma. So \( BP_\infty \langle n + 1 \rangle \langle g \rangle \) is monic. Continuing inductively, we see that \( BP_\infty \langle n \rangle \langle g \rangle \) is monic for \( N \geq n \). For \( N \) sufficiently large, \( BP_\infty(BP_\infty(g)) \) is \( BP_\infty \langle n \rangle \langle g \rangle \). Thus \( 0 = BP_\infty \langle n \rangle \langle f \rangle(1_\infty) = BP_\infty \langle f \rangle(1_\infty) = \eta_\infty \circ f \). Since \( \eta_\infty \) is a monomorphism, \( [f] = 0 \). We conclude that when \( \hom \dim_{BP_\infty} BP_\infty(X) \leq n \), the kernel of \( \eta_n \) is trivial.

Remark 6.4. Recall from (2.7) that \( BP_\infty(BP_\infty) \) is equivalent to \( G \), the Adams–Anderson–Meiselman summand of connective \( k \)-theory localized at the prime \( p \). It was shown in Proposition 6.2 of [17] that the \( n = 1 \) case of Theorem 6.1 is equivalent to: the Hurewicz homomorphism

\[
\eta_1 : BP_\infty(X) \to G_\infty(BP \wedge X)
\]
is a $Z(p)$ split monomorphism for a finite complex $X$ with $\text{hom dim}_{BP_*} BP_*(X) = 0$. It is well known [4] and elementary to show that this last statement is equivalent to the Hattori form of the Stong–Hattori theorem [4; 15; 16; 28; 34].

**Corollary 6.5.** Let $f: X \to Y$ be a stable map of finite complexes. If $\text{hom dim}_{BP_*} BP_*(X) \leq n$ and if $BP(n)_*(f): BP(n)_*(X) \to BP(n)_*(Y)$ is monic, then $BP_*(f): BP_*(X) \to BP_*(Y)$ is also monic.

**Proof.** The corollary follows immediately from the commutative diagram (6.6)

\[
\begin{array}{ccc}
0 & \rightarrow & BP_*(X) \\
\downarrow & & \downarrow BP_*(f) \\
BP_*(X) & \rightarrow & BP_*(Y) \\
\rightarrow \eta_n & & \rightarrow \eta_n \\
BP(n)_*(BP \wedge X) & \rightarrow & BP(n)_*(BP \wedge Y) \\
\cong & & \cong \\
0 \rightarrow BP_*(BP) \otimes_{BP_*} BP(n)_*(X) & \rightarrow & BP_*(BP) \otimes_{BP_*} BP(n)_*(Y) \\
\rightarrow BP_*(BP) \otimes_{BP_*} BP(n)_*(f) & & \rightarrow BP_*(BP) \otimes_{BP_*} BP(n)_*(g)
\end{array}
\]

**Corollary 6.7 (Conner).** Let $X$ be a finite complex. If $0 \neq \gamma \in \langle x_{n+1}, x_{n+2}, \ldots \rangle BP_*(X) \subseteq \ker p(n, \infty)$ is spherical, then $\text{horn dim}_{BP_*} BP_*(X) \geq n + 1$.

**Proof.** Let $f: S^n \to X$ be a stable map representing $\gamma$ in that $BP_*(f)_{\infty} = \gamma$. Let $g: X \to Y$ be the canonical map to the cofibre of $f$. Since $0 = p(n, \infty) \gamma = p(n, \infty) BP_*(f)_{\infty} = BP(n)_*(f)_{\infty} = BP(n)_*(g)_{\infty}$ is monic. Since $0 \neq \gamma = BP_*(f)_{\infty} \in \ker BP_*(g)$, $\text{hom dim}_{BP_*} BP_*(X) \geq n$ by (6.5).

**Conjecture 6.8.** If $S^n \xrightarrow{f} X \xrightarrow{g} Y$ is a cofibration sequence of finite complexes and if $\text{hom dim}_{BP_*} BP_*(X) \leq n$, then $\text{hom dim}_{BP_*} BP_*(Y) \leq n + 1$.

**Remark 6.9.** We once advertised (6.8) as being proved. P. E. Conner observed that (6.7) is a corollary of (3.13) and (6.8). Although we found a gap in our argument for (6.8), Conner's observation motivated our proof of Theorem 6.1.

The main result of [18] is that if $X$ is a finite complex with $k$-skeleton $X^k$, then $\text{hom dim}_{MU_*} MU_*(X) \leq n$ implies $\text{hom dim}_{MU_*} MU_*(X^k) \leq n$ provided that $n = 0, 1, 2$. Then $n - 2$ version follows from Proposition 6.10 by skeletal induction. The reader is invited to construct his own proof of the $n = 1$ version.

**Proposition 6.10.** Let $\vee S^m \xrightarrow{f} X \xrightarrow{g} Y$ be a cofibration of finite complexes. Suppose the cellular dimension of $X$ is at most $m$. If $\text{hom dim}_{BP_*} BP_*(Y) \leq 2$, then $\text{hom dim}_{BP_*} BP_*(X) \leq 2$.

**Proof.** Suppose $\text{hom dim}_{BP_*} BP_*(X) > 2$; so there is an element $0 \neq a \in BP(2)_*(X)$ with $x_2^* a = 0$ (3.13). We may write $a = x_2^* b$ with $0 \neq \rho(1, 2)(b) \in BP(1)_*(X)$. $b$ is a $T_2$
torsion element and thus is a $T_1$ torsion element (3.2). So $\rho(1, 2)(b)$ is $T_1$ torsion. Consider the $BP$ Bockstein sequence

$$H_{s+2p-1}(X; Z_{(p)}) \rightarrow \Delta_1 BP\langle 1 \rangle_{s}(X) \rightarrow x_1 BP\langle 1 \rangle_{s+2p-2}(X).$$

Since $H_{s+2p-1}(X; Z_{(p)}) = 0$ for $s + 2p - 1 > m$, $x_1$ multiplication is monic in dimensions greater than $m - 2p + 1$. We conclude that the dimension of $b$ is well below $m$ and thus $0 \neq g_{s}(b) \in BP\langle 2 \rangle_{s}(Y)$. Since $b$ is $T_2$ torsion, this would give $T_2$ torsion in $BP\langle 2 \rangle_{s}(Y)$ contradicting the hypothesis that $\text{hom dim}_{BP} BP\langle 2 \rangle_{s}(Y) \leq 2$ (3.13).

**Question 6.11.** If $0 \neq y \in BP\langle n \rangle_{s}(X)$ is a $T_n$ torsion element, can $s + 1$ be greater than the cellular dimension of $X$?

**Acknowledgements** This paper seems to be a natural outgrowth of our respective theses written at Virginia and M.I.T. and directed by E. E. Floyd and F. P. Peterson. We would also like to thank P. E. Conner and L. Smith who have been most generous in their observations and encouragement. Gracias to the II E.L.A.M. held at the Centro de Investigación del I.P.N. in Mexico City, 1971, where discussions leading to this paper began.

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