THE \(\Omega\)-SPECTRUM FOR BROWN-PETerson COHOMOLOGY PART II.

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**Introduction.** Let \(BP\) denote the spectrum for the Brown-Peterson cohomology theory, \(BP^*(-)\). [2, 5, 12] We have \(BP^k(X) \approx [X,BP_k]\) where \(BP = \{BP_k\}\) as an \(\Omega\)-spectrum, i.e. \(\Omega BP_k \approx BP_{k-1}\). [4] In Part I [20] we determined the structure of the cohomology of \(BP_k\). In this part we study the homotopy type of \(BP_k\).

The structure of each \(BP_k\) is very nice and gives some insight into the cohomology theory. In particular, using it, we obtain a new proof of Quillen’s theorem that \(BP^*(X)\) is generated by non-negative degree elements as a module over \(BP^*(S^0)\). [11] \((X\) is a pointed finite CW complex.)

Let \(\mathbb{Z}(p)\) be the integers localized at \(p\), the prime associated with \(BP\). We explicitly construct spaces \(Y_k\) which are the smallest possible \(k\)–connected \(H\)-spaces with \(\pi_*\) and \(H_*\) free over \(\mathbb{Z}(p)\). The \(Y_k\) are the building blocks for \(BP_n\), i.e., \(BP_n \approx \prod Y_k\). In fact, one of our main theorems states that for any \(H\)-space \(X\) with \(\pi_*\) and \(H_*\) free over \(\mathbb{Z}(p)\), then \(X \cong \prod Y_k\). (This is not as \(H\)-spaces, see section 6.) To understand the spaces \(Y_k\) we need a sequence of homology theories:

\[
BP^*(X) \approx BP^\infty(X) \rightarrow \cdots \rightarrow BP^{n+1}(X) \rightarrow BP^{n}(X) \rightarrow \cdots \rightarrow BP^0(X) = H_*(X, \mathbb{Z}(p))
\]

These are constructed using Sullivan’s theory of manifolds with singularities. \(BP^*(S^0) \approx \mathbb{Z}(p)[x_1, x_2, \ldots]\) with degree of \(x_i = 2(p^i - 1)\). \(BP^n(X) \approx \mathbb{Z}(p)[x_1, x_2, \ldots]\) as a graded group. Let \(BP^n = \{BP^n\}\) be the \(\Omega\)-spectrum for \(BP^n(X)\). For \(k > 2(p^{n-1} + \cdots + p + 1)\), the space \(BP^n_k\) cannot be broken down as a product \(BP^n_k \approx X \times X\) with both \(X\) and \(Y\) non-trivial. For \(k < 2(p^n + \cdots + p + 1)\), \(H^*(BP^n_k, \mathbb{Z}(p))\) has no torsion. So, for \(k\) between these two numbers we get \(Y_k \approx BP^n_k\).

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Main Theorem. For $2(p^n - 1 + \cdots + p + 1) < k < 2(p^n + \cdots + p + 1)$

$$BP_k \cong BP(n)_k \times \prod_{j > n} BP(j)_{k + 2(p^j - 1)}$$

and cannot be broken down further.

The proof of this theorem exploits the fact from [20] that the $Z_{(p)}$ cohomology of $BP_k$ has no torsion.

We begin by constructing the theories $BP(n)_*(\cdot)$. In section 2 we review what we need about Postnikov systems. Section 3 is devoted to preliminary necessities for the proof of the main theorems in section 4. Then we state the main results and prove Quillen’s theorem (section 5) and a general decomposition theorem for spaces which are $p$-torsion free and $H$-spaces when localized at $p$. (section 6)

In a paper with Dave Johnson these results are applied to study the homological dimension of $BP^*(X)$ over $BP^*(S^n)$. [21]

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1. Construction of $BP(n)_*$.

This section deals with Sullivan’s theory of manifolds with singularities. [19] The approach we take is due to Nils Baas. This section is not intended to be an exposition on the Baas-Sullivan theory, for we only wish to use it to construct certain specific homology theories, the general case being covered in detail in [3]. Even the definitions we give will be missing major ingredients, in all cases we refer to [3].

If we dealt with the case of one singularity, $P$, then a manifold with singularity $P$ would be a space $V = N \cup_{P \times M} cP \times M$ where $N$ is a manifold with $\partial N = P \times M$ and $cP$ is the cone on $P$. One can make a bordism group of a space using such objects in place of manifolds. An element of the bordism would be represented by a map $f: V \to X$. So, as far as bordism is concerned, one might just as well consider only the manifold $N$ and insist that maps $f: N \to X$, when restricted to $\partial N = P \times M$, factor through the projection $P \times M \to M$. This is the approach Baas takes. When more than one singularity is considered, the definitions become quite technical. From [3]

Definition. $V$ is a closed decomposed manifold if there exist submanifolds $\partial_1 V, \ldots, \partial_n V$ such that $\partial V = \partial_1 V \cup \cdots \cup \partial_n V$ where union means identification along common part of boundary such that $\partial(\partial_i V) = (\partial_1 V \cap \partial_i V)$
$U \cdots (\partial_{i-1} V \cap \partial_i V) U \varnothing U \cdots (\partial_n V \cap \partial_i V)$, which gives $\partial_i V$ the structure of a decomposed manifold. Continue, defining $\partial_k (\partial_i (\partial_j V))$, etc.

Let $S^n = \{P_1, P_2, \ldots, P_n\}$ be a fixed class of manifolds. Very loosely, $A$ is a closed manifold of singularity type $S^n$ if for each subset $\omega \subset \{1, 2, \ldots, n\}$ there is a decomposed manifold $A(\omega)$ such that $A(\emptyset) = A$, $A(\omega) = A(\omega, i) \times P_i$ if $i \in \omega$, $\partial_i A(\omega) = \emptyset$ if $i \notin \omega$. A singular $S^n$ manifold in $X$ is a map $g: A \to X$ such that $g|\partial_i A(\omega) = A(\omega, i) \times P_i$ factors through the projection $A(\omega, i) \times P_i \to A(\omega, i)$.

More generally, singular manifolds with boundary, singular manifolds in a pair, and a concept of bordism are all defined. (Rigorously) These bordism groups are shown to give generalized homology theories, $MS^n_*(\cdot)$. One of the most important aspects of these theories is the relationship between $MS^n_*(\cdot)$ and $MS^{n+1}_*(\cdot)$. This will be a major tool throughout the paper. There is an exact sequence

$$\xy
\xymatrix{MS^n_*(X) & & MS^n_*(X) \ar[l]_{\beta} \ar[d]^\delta \ar[r]^\gamma & MS^{n+1}_*(X)}
\endxy$$

The product of an $S^n$ manifold with a closed manifold $N$ gives an $S^n$ manifold by: $(N \times A)(\omega) = N \times A(\omega)$. On a representative element $A \to X$, $\beta$ is $P_{n+1} \times A \\ A \to X$. Any $S^n$ manifold $A$ can be considered as an $S^{n+1}$ manifold by setting $A(\omega, n+1) = \emptyset$. So $\gamma(A \to X) = (A \to X)$. For an $S^{n+1}$ manifold $A$ we see that $A(n+1)$ is an $S^n$ manifold, so $\delta(f: A \to X) = f|A(n+1) \to X$. The degrees of these maps are: degree $\beta =$ dimension $P_{n+1}$, degree $\gamma = 0$, degree $\delta = -\text{dimension } P_{n+1} - 1$. In our one singularity example, $\partial N = P \times M$, $\delta$ just restricts to $M$. Baas of course defines these maps rigorously, shows they are well defined and proves the exactness theorem.

Above we remarked that the product of a manifold and an $S^n$ manifold is again an $S^n$ manifold. This gives us a map, $MS^0_*(X) \otimes MS^n_*(Y) \to MS^n_*(X \times Y)$. This is precisely the condition that tells us the spectrum associated with $MS^n_*(\cdot)$ is a module spectrum over the spectrum for the standard bordism theory $MS^0_*(\cdot)$. Further, $MS^n_*(X)$ is a module over $MS^0_*(S^0)$ and the above maps, $\beta$, $\gamma$, $\delta$ are all $MS^0_*(S^0)$ module maps.

We now get on to our applications. All manifolds considered above could be taken with some extra structure, and we assume them all to be $U$ manifolds. So $MS^0_*(\cdot)$ is $MU_*(\cdot)$ the standard complex bordism homology theory for finite complexes. Now $MU_*(S^0) = \pi^n_*(MU) = Z[x_2, \ldots, x_{2i}, \ldots]$ where degree $x_{2i} = 2i$. We choose a representative manifold $P_i$ for $x_{2i}$. Fix a prime $p$. $S(n, m) =$
\{P_i | i < m, i \neq p^j - 1, j < n\}. Then by all of the above, we have a homology theory \(MUS(n, m)_* (\cdot)\) made from \(U\) manifolds with singularity type \(S(n, m)\).

For large \(m\) we have an exact sequence:

\[
\begin{array}{c}
MUS(n, m)_* (X) \\
\xrightarrow{\times p^{n-1}} \\
\xleftarrow{\delta} \\
MUS(n-1, m)_* (X) \\
\xrightarrow{\gamma} \\
MUS(n, m)_* (X)
\end{array}
\]

From these exact sequences and the homotopy of \(MU\) we see that \(MUS(n, m)_* (S^0) = \pi_{*}^{S_*} (MU) / [S(n, m)]\) where \([S(n, m)]\) is the ideal generated by \(S(n, m)\). We define the homology theory \(MUS(n)_* (\cdot) = \lim (m \to \infty) MUS(n, m)_* (\cdot)\). \(MUS(n)_* (\cdot) \otimes Z(p)\) is a homology theory which we will denote by \(BP\langle n \rangle_\ast (\cdot)\) and the corresponding spectrum by \(BP\langle n \rangle\). The reason for the notation is that if \(BP \to MU\langle p \rangle\) is Quillen’s map ([12]), then \(BP \to MU\langle p \rangle \to BP\langle \infty \rangle\) clearly gives an isomorphism on homotopy and so \(BP \cong BP\langle \infty \rangle\). Thus \(BP\langle n \rangle\) is a module spectrum over \(BP\) and we have:

\[
\begin{array}{c}
BP\langle n \rangle_\ast (X) \\
\xleftarrow{\delta} \\
BP\langle n-1 \rangle_\ast (X) \\
\xrightarrow{\gamma} \\
BP\langle n \rangle_\ast (X)
\end{array}
\]

with degree of \(\beta = 2(p^n - 1)\), degree \(\gamma = 0\), degree \(\delta = -2p^n + 1\). \(BP_* (S^0) = Z(p)[x_{2(p-1)}, x_{2(p^2-1)}, \ldots]\). \(BP\langle n \rangle_* = BP\langle n \rangle_* (S^0) / [x_{2(p^i-1)} | i > n]\) as a module over \(BP_*\). \(BP_*\) acts on \(BP\langle n \rangle_\ast (X)\), it is known that \(x_{2(p^i-1)}\) acts trivially for \(i > n\).

Every spectrum can be represented as an \(\Omega\)-spectrum. [4] Let \(BP\langle n \rangle = \{BP\langle n \rangle_k\}\) be the \(\Omega\)-spectra, i.e. \(\Omega BP\langle n \rangle_k \cong BP\langle n \rangle_{k-1}\) and \(BP\langle n \rangle_k\) is \(k-1\) connected for \(k > 0\). This means that \(BP\langle n \rangle^k (X) = [X, BP\langle n \rangle_k]\) where \(BP\langle n \rangle^k (\cdot)\) is the cohomology theory given by \(BP\langle n \rangle\).

The theories \(BP\langle n \rangle\) are independent of choice of manifolds \(P_i\) representing \(x_{2i}\) but seemingly dependent on the choice of generators \(x_{2(p^i-1)}\) chosen for \(\pi_{*}^{S_*} (MU)\). However, the results we obtain are independent of the choice of even these generators because the spaces \(BP\langle n \rangle_k\) for different choices become homotopy equivalent when \(k\) is small enough. In addition, in [21] we show that \(BP\langle 1 \rangle\) is independent of choice of \(x_{2i}\). In fact, \(BP\langle 1 \rangle\) is just the irreducible part of connective \(K\)-theory when localized at \(p\).
We now permanently reindex the $x_{2(p^i-1)}$ to $x_i$ with degree $2(p^i-1)$. From 1.1 we have a split exact sequence:

$$0 \xrightarrow{x_n} BP(n)_* \xrightarrow{} BP(n)_* \xrightarrow{} BP(n-1)_* \xrightarrow{} 0$$  \hspace{1cm} (1.2)

$BP(n)_* = Z_{(p)}[x_1, \ldots, x_n]$ as a group. Again, from 1.1 for finite complexes we get a cofibration ([1]):

$$S^{2i}BP(n) \xrightarrow{} BP(n) \xrightarrow{} BP(n-1)$$  \hspace{1cm} (1.3)

For the spaces in the $\Omega$-spectrum this becomes a fibration:

$$BP(n)_{k+j} \xrightarrow{} BP(n)_k \xrightarrow{} BP(n-1)_{k+j}$$  \hspace{1cm} (1.4)

If $M$ is a graded module let $s^kM$ be the graded module $(s^kM)_{k+q} = M_q$. Then,

$$\pi_*(BP(n)_k) = s^k(BP(n)_*) \quad k \geq 0$$  \hspace{1cm} (1.5)

From 1.3 we have an exact sequence:

$$H^*(BP(n)) \xrightarrow{\beta^*} H^*(BP(n)) \xrightarrow{\gamma^*} H^*(BP(n-1))$$  \hspace{1cm} (1.6)

For most of the paper, unless otherwise noted, all coefficient groups will be $Z_p$ where $p$ is the fixed prime associated with the $BP(n)$. Let $A$ be the mod$p$ Steenrod algebra and $Q_i$ the Milnor elements. [9]

**Proposition 1.7.** $H^*(BP(n)) \cong A/A(Q_0, Q_1, \ldots, Q_n) = A_n$

**Note.** Baas and Madsen have a more general result which includes this, however, as this special case has a much more elementary proof we give it here.

**Proof.** $\pi^S_*(BP(0)) = Z_{(p)}$, so $BP(0) = K(Z_{(p)})$ and $H^*(BP(0)) = A/A(Q_0)$. We prove the result by induction on $n$ using 1.6. Let 1 denote the lowest dimensional class of each spectrum, then $\gamma^*(1) = 1$. Assume $H^*(BP(n-$
1>) = A_{n-1}. If \( \gamma^*(Q_n) = 0 \), then for dimensional reasons, \( \partial^*(1) = \lambda Q_n, 0 \neq \lambda \in \mathbb{Z}_p \). If \( a \in A_n \), then \( 0 \neq a Q_n = \partial^*(a 1) \) in \( A_{n-1} \) because \( A_{n-1} = A_n(1) \oplus A_n(Q_n 1) \). Therefore \( a 1 \neq 0 \) in \( H^*(BP<n>) \). This takes care of exactness at \( H^*(BP<n-1>) \). So now \( H^*(BP<n>) = A_n \oplus X \) with \( \beta^*: X \to X \) an isomorphism, but the degree of \( \beta^* \neq 0 \) so \( X = 0 \).

All we need now is \( Q_{n1} = 0 \in H^*(BP<n>) \). Our map \( BP \to BP<n> \) is an isomorphism on homotopy below dimension \( 2(p^{n+1} - 1) \) and therefore an isomorphism on cohomology in this range. \( H^*(BP) \cong \mathbb{A}/A(Q_0, Q_1, \ldots) \). [5] The dimension of \( Q_n \) is \( 2p^n - 1 \) so \( Q_{n1} = 0 \).

2. Postnikov Systems.

We collect here the results we need about Postnikov systems. We assume \( X \) is a simply connected CW complex. We start with the standard diagram:

\[
\begin{array}{cccccc}
  &   &   &   & X^n &   &   \\
  &   &   &   & \downarrow &   &   \\
  &   &   & X^n & \rho_n &   &   \\
  &   &   & \uparrow & g_n & \downarrow &   \\
  &   & X & \rho_{n-1} &   &   \\
  &   & K(\pi_n(X), n+1) &   &   &   \\
  &   &   & \downarrow &   &   \\
  &   & X^1 = \text{point} &   &   &   \\
\end{array}
\]

2.1. Definition and existence. [16] A Postnikov system for \( X \) is a sequence of spaces \( \{X^n\} \) and maps, \( \{g_n: X^n \to X^{n-1}\} \), \( \{\rho_n: X \to X^n\} \) such that \( \rho_{n-1} \cong g_n \rho_n \) and the fibre of \( g_n \) is \( K(\pi_n(X), n) \), the Eilenberg-MacLane space. The fibration \( g_n: X^n \to X^{n-1} \) is induced by a map \( k_n: X^n \to K(\pi_n(X), n+1) \) from the path space of \( K(\pi_n(X), n+1) \). Thus \( k_n g_n \equiv 0 \) and \( k_n \in H^{n+1}(X^n, \pi_n(X)) \). \( k_n \) is called the \( n \)-th \( k \)-invariant of \( X \). Postnikov systems for simply connected CW complexes always exist and \( (\rho_n)_*: \pi_k(X) \to \pi_k(X^n) \) is an isomorphism for \( k \leq n \) and \( \pi_k(X^n) = 0 \) for \( k > n \).

2.2. Induced maps. [8] Given \( f: X \to Y \) then we have \( \{f^n: X^n \to Y^n\} \)
such that \( f^{n-1} \cdot g_n(X) \cong g_n(Y) \cdot f^n \), \( f^n : \rho_n(X) \cong \rho_n(Y) \cdot f \) and \( f (k_n(X)) = (f^{n-1} \cdot k_n(Y)) \).

2.3. Loop spaces. The Postnikov system for \( \Omega X \) is given by: \( (\Omega X)^n = \Omega X^{n+1} \), \( \rho_n(\Omega X) = \Omega \rho_{n+1}(X) \), \( g_n(\Omega X) = \Omega g_{n+1}(X) \), \( k_n(\Omega X) = \Omega k_{n+1}(X) \), so \( k_n(\Omega X) = s^* (k_{n+1}(X)) \in H^{n+1}(\Omega X, \pi_n(\Omega X)) \) where \( s^* \) is the cohomology suspension defined by \( \delta^{-1} \cdot p^* \).

\[
H^*(\Omega X, G) \xrightarrow{\delta} H^{*+1}(PX, \Omega X, G) \xleftarrow{p^*} H^{*+1}(X, pt, G)
\]

\( PX \) is the path space fibration over \( X \).

2.4. Product spaces. A Postnikov system for \( X \times Y \) is given by \( \{X^n \times Y^n\} \) with \( k \)-invariants \( \{k_n(X) \times k_n(Y)\} \).

2.5. \( H \)-spaces. [8] If \( X \) is an \( H \)-space, then each \( X^n \) is an \( H \)-space, \( \rho_n \) and \( g_n \) are maps of \( H \)-spaces and \( k_n \in H^{n+1}(X^{n-1}, \pi_n(X)) \) is torsion and is primitive in the Hopf algebra structure induced on \( H^* \) by the multiplication in \( X^{n-1} \). Also, if \( X^{n-1} \) is an \( H \)-space and \( k_n \) is primitive, then \( X^n \) is an \( H \)-space. If all \( k \)-invariants are primitive, then \( X \) is an \( H \)-space.

2.6. Obstruction theory. [16] If \( Y \) is CW, and we have \( f_{n-1} : Y \to X^n \), then \( f_{n-1} \) lifts to \( f_n : Y \to X^n \) iff \( (f_{n-1})^*(k_n(X)) = 0 \in H^{n+1}(Y, \pi_n(X)) \). If there exist maps \( \{f_n : Y \to X^n\} \) such that \( g_n(X) \cdot f_n \approx f_{n-1} \) then there exists \( f : Y \to X \) with \( \rho_n(X) \cdot f \approx f_n \).

2.7. Construction of spaces. [16] Given a sequence of fibrations \( g_n : X^n \to X^{n-1} \) with fibre \( K(\pi_n, n) \) and \( X^1 = pt \), then there exists a CW complex \( X \) and maps \( \rho_n : X \to X^n \) such that \( \{X^n\} \) is a Postnikov system for \( X \).

2.8. Independent \( k \)-invariants. Assume for the rest of this section that \( \pi_s(X) \otimes Z(p) \) is free over \( Z(p) \) and the \( k \)-invariants \( k_n(X) \) are torsion elements. This will always be the case in our applications. From the Serre spectral sequence of a fibration we obtain the following natural ladder of exact sequences:

\[
\begin{array}{cccc}
0 & \to & H^s(X^{s-1}, Z(p)) & \xrightarrow{(g_s)^*} & H^s(X^s, Z(p)) & \to & H^s(K(\pi_s(X), s), Z(p)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^s(X^{s-1}) & \xrightarrow{(g_s)^*} & H^s(X^s) & \to & H^s(K(\pi_s(X), s)) \\
\end{array}
\]
2.9.
\[
\begin{array}{c}
\tau: H^{s+1}(X^{s-1}, Z_{(p)}) \\
\downarrow \\
\tau^*: H^{s+1}(X^{s-1}) \\
\end{array}
\xrightarrow{(\rho)^*} \begin{array}{c}
H^{s+1}(X^s, Z_{(p)}) \\
\downarrow \\
H^{s+1}(X^s) \\
\end{array} \rightarrow 0
\]

\(\tau\) is the transgression. Also we obtain

2.10. \(H^k(X^s, G) \cong H^k(X, G)\) for \(k < s\). In the dimension of our ladder, the transgression is related to the \(k\)-invariant map \(k_s\) by \(\tau \cdot s^* = k_s^*\). This motivates the following definitions.

For \(x \in H^s(K(\pi_s(X), s), Z_{(p)})\), a free generator, \(\tau(x)\) will be called a \(k\)-invariant of \(X\). If \(\tau(x) = 0\), it is called dependent. The \(k\)-invariant \(\tau(x)\) is independent and hits a \(p\)-torsion generator if and only if \(\rho \cdot \tau(x) = \tilde{\tau} \cdot \rho(x) \neq 0\) where \(\rho\) is the mod \(p\) reduction. If the \(k\)-invariants, \(\tau(x)\), of \(\Omega X\), hit \(p\)-torsion generators, then there is a \(y\) with \(s^*(y) = x\), and so \(s^*(\tau(y)) = \tau(x)\) showing that the \(k\)-invariants \(\tau(y)\) of \(X\) also hit \(p\)-torsion generators. (Remember that we have restricted ourselves to spaces with torsion \(k\)-invariants.) If \(H^*(X, Z_{(p)})\) has no \(p\) torsion, then all \(p\) torsion generators of \(H^{s+1}(X^{s-1}, Z_{(p)})\) are hit by \(k\)-invariants. This is true because the coker \(\tau \cong H^{s+1}(X^s, Z_{(p)}) \subset H^{s+1}(X^{s+1}, Z_{(p)}) \cong H^{s+1}(X, Z_{(p)})\) which is free, all by 2.9 and 2.10.

2.11. Localization. [18] Usually we will work with localized spaces, i.e. spaces with \(\pi_*(X)\) a \(Z_{(p)}\) module. For simply connected spaces or \(H\)-spaces, the localization \(X_{(p)}\) and a mod \(p\) equivalence \(X \rightarrow X_{(p)}\) can be built by 2.7 using \(\pi_*(X) \otimes Z_{(p)}\) for homotopy groups and \(k_n(X) \otimes Z_{(p)}\) as the \(k\)-invariants. We get that

\[H_*(X, Z) \otimes Z_{(p)} \cong H_*(X, Z_{(p)}) \cong H_*(X_{(p)}, Z_{(p)}) \cong H_*(X_{(p)}, Z).\]

2.12. Irreducible spaces. If a space cannot be written as a non-trivial product of spaces it will be called irreducible (indecomposable). If \(X\) is connected and \(\Omega X\) is irreducible, then \(X\) must also be irreducible. If \(X\) is a localized space with \(\pi_*(X)\) free (over \(Z_{(p)}\)) then if \(X^{s-1}\) in the Postnikov system for \(X\) is irreducible and all of the \(s\) \(k\)-invariants are independent, then \(X^s\) is also irreducible.

3. The Map.

Before we can prove the main theorem,

\[BP_k \cong BP\langle n \rangle_k \times \prod_{j > n} BP\langle j \rangle_{k + 2(p^j - 1)}\]
for \( k < 2(p^n + \cdots + p + 1) \), we need the maps \( BP_k \to BP\langle j \rangle_{k+2(p^l-1)} \). The natural transformation \( BP_n(\cdot) \to BP\langle n \rangle (\cdot) \) gives us the map \( BP_k \to BP\langle n \rangle_k \) which is onto in homotopy. If we obtain the map \( BP_k \to BP\langle j \rangle_{k+2(p^l-1)} \) for \( k = 2(p^l-1 + \cdots + p + 1) \) then we have it for all \( k < 2(p^l-1 + \cdots + p + 1) \) by taking the loop map. We can then combine these maps to give a map

\[
BP_k \to BP\langle n \rangle_k \times \prod_{j > n} BP\langle j \rangle_{k+2(p^l-1)} \quad \text{for} \quad k < 2(p^n + \cdots + p + 1).
\]

We fix \( k = 2(p^l-1 + \cdots + p + 1) \) and construct a map \( BP_k \to BP\langle j \rangle_{k+2(p^l-1)} \) by the following series of lemmas.

**Lemma 3.1.** There is an element \( x_i \in H^{k+2(p^l-1)}(BP\langle j \rangle_k, Z(p)) \) such that \( x_i : BP\langle j \rangle_k \to K(Z(p), k+2(p^l-1)) \) is onto in homotopy, \( k < 2(p^l-1 + \cdots + p + 1) \).

Before proceeding, we need to state a lemma which we will prove later. Let \( i_k \) be the generator of \( H^k(BP\langle j-1 \rangle_k) \).

**Lemma 3.2.** For \( k > 2(p^l-1 + \cdots + p + 1) \), \( Q_i i_k \neq 0 \) in \( H^*(BP\langle j-1 \rangle_k) \). For \( k = 2(p^l-1 + \cdots + p + 1) \), \( H^i(BP\langle j-1 \rangle_k) = 0 \) for \( i = k + 2p^l-1 = \text{dimension} \), \( Q_i i_k = p^l + 1 \).

**Proof of 3.1.** We go to the fibration 1.4.

\[
\begin{array}{cc}
BP\langle j \rangle_s & \xrightarrow{\beta} BP\langle j \rangle_k \\
s = k+2(p^l-1) & \xrightarrow{\gamma} BP\langle j-1 \rangle_k \\
k = 2(p^l-1 + \cdots + p + 1) & \end{array}
\]  (3.3)

\( BP\langle j \rangle_s \) is \( s-1 \) connected and \( \pi_s(BP\langle j \rangle_s) \cong H^s(BP\langle j \rangle_s, Z(p)) \cong Z(p) \).

To show \( \beta^* \) is onto in dimension \( s \) we look at the Serre spectral sequence for the fibration 3.3. In this range we have the Serre exact sequence:

\[
H^s(BP\langle j \rangle_k, Z(p)) \xrightarrow{\beta^*} H^s(BP\langle j \rangle_s, Z(p)) \to H^{s+1}(BP\langle j-1 \rangle_k, Z(p)) \]  (3.4)

We have \( k = 2(p^l-1 + \cdots + p + 1) \) and so \( s+1 = k+2(p^l-1)+1 \). By 3.2 and the numbers we are using, the last term is zero and so \( \beta^* \) is onto. If \( x_i : H^s(BP\langle j \rangle_k, Z(p)) \) is such that \( \beta^*(x_i) \) is the generator, and \( S^s \to BP\langle j \rangle_s \) represents \( 1 \in \pi_s(BP\langle j \rangle_s) = Z(p) \), then the composition \( S^s \to BP\langle j \rangle_s \to BP\langle j \rangle_k \) induces an isomorphism on \( H^s \), so therefore \( x_i \) is onto in homotopy.
LEMMA 3.5. There is a map \( f_j : BP_k \to BP^j_k + 2(p^i - 1) \) for \( k < 2(p^i - 1 + \cdots + p + 1) \), such that \( (f_j)_\# \) is onto, \( (k > 2(p^i - 1)) \)

\[
(f_j)_\# : \pi_{k + 2(p^i - 1)}(BP_k) \to \pi_{k + 2(p^i - 1)}(BP^j_k) \cong Z(p)
\]

Proof. It is enough to prove this for \( k = 2(p^i - 1 + \cdots + p + 1) \). We have a map \( BP_k \to BP^j_k \) from Lemma 3.1. Each of these maps is onto in homotopy so the composite is too. \( K(Z(p), k + 2(p^i - 1)) \) is the first non-trivial term of the Postnikov system for \( BP^j_k + 2(p^i - 1) \). We know that the \( k \)-invariants of this space are torsion by 2.5 and that its homotopy is free over \( Z(p) \) by construction. (1.5) The main theorem of [20] gives us that \( H^*(BP_k, Z(p)) \) has no torsion. Obstructions to lifting the map to a map of the type we want are therefore torsion elements in \( H^{q + 1}(BP_k, \pi_q(BP^j_k)) \), (2.6) which has no torsion. Therefore we see that we can lift the map.

COROLLARY 3.6. For \( k < 2(p^i + \cdots + p + 1) \), there is a map \( BP_k \to BP^j_k \times \prod_{j > n} BP^j_k + 2(p^i - 1) \) which composed with projections is onto in homotopy for \( \pi_*(BP^j_k) \) and \( \pi_{k + 2(p^i - 1)}(BP^j_k + 2(p^i - 1)) \).

Before proving 3.2 we will make an observation which we need in the next section.

Consider the map \( \beta : BP^j_k \to BP^j_k, s = k + 2(p^i - 1) \). We have \( \beta \# : K(Z(p), s) \to K(\pi_*(BP^j_k), s). (\beta \#)_* \) is onto in \( Z(p) \) cohomology. Pick a generator \( x \in H^s(K(\pi_*(BP^j_k), s), Z(p)) \) such that \( (\beta \#)_*(x) \) is a generator. We wish to study the \( k \)-invariant \( \tau(x) \). Above we showed that for \( k < 2(p^i - 1 + \cdots + p + 1) \) there was such a \( k \)-invariant which was dependent. Here we wish to show the following lemma.

LEMMA 3.7. For \( k > 2(p^i - 1 + \cdots + p + 1) \), the above \( k \)-invariant \( \tau(x) \) is independent and hits a \( p \)-torsion generator.

Proof. Using the naturality of the mod \( p \) version of 2.9 we have:

\[
\begin{align*}
0 & \longrightarrow H^s(K(Z(p), s)) & \cong & \quad & H^s(K(Z(p), s)) \\
\beta^* & \longrightarrow & \beta^* & \cong & \beta^* \\
\cong & \quad & \cong & \quad & \cong \\
H^s(K(Z(p), s)) & \longrightarrow & H^s(K(\pi_*(BP^j_k), s)) & \longrightarrow & 0 \\
\end{align*}
\]

\[
\begin{align*}
H^s(BP^j_k) & \cong H^s(BP^j_k) \\
\end{align*}
\]

(3.8)
As in 2.8, \( \tau(x) \) is independent and hits a \( p \)-torsion generator iff \( \tilde{\tau}(\rho(x)) \neq 0 \), \( \rho \) the mod \( p \) reduction. Because \((\beta_\ast)\ast(x)\) is a generator, this is equivalent to \( \beta^\ast \) not being onto in 3.8. Again we go to the Serre exact sequence for 1.4.

\[
H^s(BP\langle i \rangle_k) \xrightarrow{\beta^\ast} H^s(BP\langle i \rangle_s) \xrightarrow{\tilde{\tau}} H^{s+1}(BP\langle i-j \rangle_k)
\]

(3.9)

We know from the proof of 1.7 that for \( k \) very large \( \tilde{\tau}(i_k) = \lambda Q_i k, \lambda \neq 0 \). By 3.2, for \( k > 2(p^{-1} + \cdots + p + 1) \) we know \( Q_i k \neq 0 \) so \( \tilde{\tau}(i_k) = \lambda Q_i k \neq 0 \) in this range. So, in 3.9 we see that \( \beta^\ast \) is not onto and \( \tau(x) \) for such an \( x \) is an independent \( p \)-torsion generating \( k \)-invariant.

We will need the following in our proof of 3.2.

**Lemma 3.10.** \( Q_{n+1} = \lambda p^{n} + \cdots + p + 1 \pmod{A(Q_0, \ldots, Q_n)}, \lambda \neq 0 \in \mathbb{Z}_p. \)

**Note.** For \( p = 2 \), just consider \( P_i = Sq^2 \).

**Proof.** The lowest non-zero odd dimensional element of \( A/A(Q_0, \ldots, Q_n) \) is \( Q_{n+1} \). From [9], \( Q_i = [P^{n-1}, Q_{1-i}], \) so \( Q_{n+1} = P^n Q_n - Q_n P^n = -Q_n PP^n = \cdots \) less than the dimension of \( Q_{n+1} \) and also odd, so it is zero) = \( \cdots = (-1)^{n+1}Q_0 P^1 \cdots P^{n-1} P^n \pmod{A(Q_0, \ldots, Q_n)} \). Let \( k_n = 1 + p + \cdots + p^{n-1} \), all that is left to show is: (note that \( Q_0 = \beta \))

**Claim.** \( p^k P^n = \lambda P^{k+1}, \lambda \neq 0 \in \mathbb{Z}_p. \)

**Proof.** By the Adem relations,

\[
p^k P^n = \sum_{t=0}^{k_n-1} (-1)^{k_n-1-t} p^{k_n-1-t} t(p-1)(p^n-t-1)\binom{k_n-1}{k_n-1-kt}
\]

So all we need is for the binomial coefficient to be zero mod \( p \) for \( 0 < t < k_n-1 \) and \( \neq 0 \) for \( t = 0 \). First we reindex, let \( s + t = k_n-1 \). Then \( (p-1)(p^n-t-1) = (p-1)(p^n-k_n-1+s)-1 = (p-1)p^n-(p-1)k_n-1+(p-1)s-1 \). Now \( (p-1)k_n-1 = p^{n-1}-1 \), so this is \( (p-1)p^n-p^{n-1}+1+(p-1)s-1 = (p-2) p^n+(p-1)p^{n-1}+(p-1)s \). \( k_n-1+p = k_n-pk_{n-1}+ps=1+ps \). So our coefficient is:

\[
\binom{(p-2)p^n+(p-1)p^{n-1}+(p-1)s}{1+ps}
\]

We want to show this is 0 for \( 0 \leq s < k_{n-1} \) and \( \neq 0 \) for \( s = k_{n-1} \).

From [17], if \( a = \Sigma a_i p^i, b = \Sigma b_i p^i, a_i \) and \( b_i < p \), then mod \( p \),

\[
\begin{pmatrix}
    a \\
    b
\end{pmatrix} = \Pi_i \left( a_i \begin{pmatrix}
    1 \\
    b_i
\end{pmatrix} \right).
\]
So for $s < p^{n-2}$ our binomial coefficient is \[
\binom{p-2}{0} \binom{p-1}{0} \binom{(p-1)s}{1+ps}
\]
but $(p-1)s < 1 + ps$, so it is zero for $s < p^{n-2}$. Set $s = p^{n-2} + s_1 > p^{n-2}$. We get:

\[
\begin{pmatrix}
(p-2)p^n + (p-1)p^{n-1} + (p-1)p^{n-2} + (p-1)s_1 \\
p^{n-1} + 1 + ps_1
\end{pmatrix}
\]

for $s_1 < p^{n-3}$ this is zero again. Let $s_1 = p^{n-3} + s_2$. Continue like this until we get:

\[
\begin{pmatrix}
(p-2)p^n + (p-1)p^{n-1} + \cdots + (p-1)p + (p-1)s_{n-2} \\
p^{n-1} + p^{n-2} + \cdots + p^2 + 1 + ps_{n-2}
\end{pmatrix}
\]

where $0 < s_{n-2} < 1$. For $s_{n-2} = 0$, this is zero again as it is

\[
\begin{pmatrix}
(p-2)
0
\end{pmatrix}
\begin{pmatrix}
0
(p-1)
\end{pmatrix}
\begin{pmatrix}
0
(p-1)
\end{pmatrix}
\begin{pmatrix}
0
(p-1)
\end{pmatrix}
\begin{pmatrix}
0
\end{pmatrix}
\]

For $s_{n-2} = 1$, we get

\[
\begin{pmatrix}
(p-2) & (p-1) & \cdots & (p-1) & (p-1) & 0 \\
0 & 1 & \cdots & 1 & 1 & 1
\end{pmatrix}
\]

This finishes the proof of 3.10.

**Proof of 3.2.** For large $k$, $Q_{i_k} \neq 0$ in $H^*(BP<j-1>_k)$ because $H^*(BP<j-1>) = A/A(Q_0, \ldots, Q_{j-1})$. (1.7) The Eilenberg-Moore spectral sequence [15]

\[\text{Tor}_{H^*(BP<j-1>_k)}(Z_p, Z_p) = \Rightarrow H^*(BP<j-1>_k) \quad (3.11)\]

collapses in dimensions $< pk$ and on indecomposables, $s^*: QH^*(BP<j-1>_k) \to QH^*(BP<j-1>_k)$ is an isomorphism in this range. For $k > 2(p^{i-1} + \cdots + p + 1)$, dimension $Q_{i_k} < pk$ so $Q_{i_k} \neq 0$.

For $k = 2(p^{i-1} + \cdots + p + 1)$, the $E_2$ term of 3.10 has one element in dimension $pk + 1$ (for $p = 2$, none), $s^{-1}(Q_{i_k+1})$. All $Q_{i_k+1} = 0$ for $i < j$ so by 3.10 this is $s^{-1}(\lambda \beta p^{i-1} + \cdots + p + 1)_{i_k+1}$. $s^{-1}$ corresponds to the cohomology suspension (14). $s^*(\beta p^{i-1} + \cdots + p + 1)_{i_k+1} = BP^{k/2}_k = \beta(i_k)^p = 0$ so $s^{-1}(Q_{i_k+1})$ is hit by a differential and the result follows.
4. Proofs.

In the last section we constructed a map (3.6)

\[ BP_k \longrightarrow BP\langle n \rangle_k \times \prod_{j > n} BP\langle j \rangle_{k+2(p-1)} \quad \text{for} \quad k \leq 2(p^n + \cdots + p + 1) \quad (4.1) \]

If this map is a homotopy equivalence for some \( k > 0 \) then it is a homotopy equivalence for all \( k \leq 2(p^n + \cdots + p + 1) \). To see this, look at the diagram for \( f: X \rightarrow Y \)

\[
\begin{array}{ccc}
\pi_\ast(\Omega X) & \xrightarrow{\cong} & \pi_\ast(X) \\
\downarrow \Omega f_\ast & & \downarrow f_\ast \\
\pi_\ast(\Omega Y) & \xrightarrow{\cong} & \pi_\ast(Y) \\
\end{array}
\]

If either \( \Omega f_\ast \) or \( f_\ast \) is an isomorphism then so is the other and then they are both homotopy equivalences because our spaces are the homotopy type of CW complexes.

We will prove the homotopy equivalences for the

\[ k = k_n = 2(p^{n-1} + \cdots + p + 1) + 1, \quad (k_0 = 1) \]

by induction on the Postnikov system. As a plausibility argument, as well as the fact that we need it, we prove the following lemma.

**Lemma 4.2.** The homotopy is the same on both sides of 4.1.

**Proof.** \( \pi_\ast(BP_k) \cong s^k(Z(p)[x_1, x_2, \ldots]) \).

\[
\pi_\ast(BP\langle n \rangle_k \times \prod_{j > n} BP\langle j \rangle_{k+2(p-1)}) = \pi_\ast(BP\langle n \rangle_k) \oplus \pi_\ast(BP\langle j \rangle_{k+2(p-1)})
\]

\[
= s^k(Z(p)[x_1, \ldots, x_n]) \oplus s^{k+2(p-1)}(Z(p)[x_1, \ldots, x_i]) \quad (1.5)
\]

Our isomorphism takes a \( Z(p) \) generator on the right-hand side, \( s^{k+2(p-1)}[(x_i)^k \cdots (x_j)^k] \) to \( s^k[(x_i)^k \cdots (x_{i+1})^k + 1] \).

Recall that \( k_n = 2(p^{n-1} + \cdots + p + 1) + 1 \quad (k_0 = 1) \).

**Statement** \( P(n, s) \). \( i = k_n + 2(p^l - 1) \)

\[
f_{k_n + s}(BP_{k_n})^{k_n + s} \longrightarrow (BP\langle n \rangle_k)^{k_n + s} \times \prod_{j > n} (BP\langle j \rangle_{k})^{k_n + s}
\]

is a homotopy equivalence.
4.3. Statement $P(n, s)$ implies a similar statement for any $k < k_{n+1}$ replacing $k_n$.

Statement $K(n, s)$. All $k$-invariants $\tau(x)$ in

$$H^{k_n+s+2}((BP^{n}_{k_n})^{k_n+s}, \mathbb{Z}(p))$$

are independent and hit $p$-torsion generators. (see 2.8)

4.4. $K(n, s)$ implies that all $k$-invariants $\tau(x)$ in $H^{k+s+2}((BP^{n}_{k})^{k+s}, \mathbb{Z}(p))$ are independent and hit $p$-torsion generators for $k > k_n$.

Statement A.

$$P(n, s) \quad s \leq m$$

$$K(n, s) \quad s < m$$

Statement B.

(1) $K(n+j, s) \quad s < m \quad j > 0$

(2) $P(n, m) \quad m \geq 0$

$$= > P(n, m+1)$$

4.5. Now, to get things started, observe that statement $P(n, 0)$ is true for all $n$ as it just reduces to $K(Z(p), k_n) \rightarrow K(Z(p), k_n)$. Also, statement $K(0, s)$ is trivially true for all $s$ because $BP^{0}_{k_n} = 1$ is just the circle localized at $p$ and has no $k$-invariants.

Lemma 4.6. Statements A and B imply statements $P(n, s)$ and $K(n, s)$ for all $n$ and $s$.

Proof. Claim $(t)$. (a) $P(n, m)$ is true for $m \leq t$, all $n$.

(b) $K(n, m)$ is true for $m < t$, all $n$.

Claim $(t)$ is true for $t = 0$ by 4.5. We will show claim $(t) = > claim (t+1)$. By 4.5 we know $K(0, t)$ is true, applying statement $A$ $n$ times we have $K(n, t)$, therefore we have $K(n, t)$ for all $n$ giving us $b$) of claim $(t+1)$. Now, applying statement $B$ we obtain $P(n, t+1)$ for all $n$. This proves claim $(t+1)$, so, by induction, claim $(t)$ is true for all $t$ and we are done.

Now we will prove statements $A$ and $B$. In the next section we will explore some of the consequences of $P(n, s)$ and $K(n, s)$. 
Proof of statement A. Consider the fibration 1.4

$$\beta: BP\langle n+1 \rangle_i \rightarrow BP\langle n+1 \rangle_k$$

$$\gamma: k = k_{n+1}$$

$$i = k_{n+1} + 2(p^{n+1} - 1)$$

BP\langle n \rangle_k$$

and the induced maps on the Postnikov systems:

$$\beta_{q-1}\downarrow$$

$$\gamma_{q-1}\downarrow$$

$$\beta_{#}\downarrow$$

$$\gamma_{#}\downarrow$$

$$k$$

$$k$$

$$K(\pi_q(BP\langle n+1 \rangle_i), q+1) \rightarrow K(\pi_q(BP\langle n+1 \rangle_k), q+1) \rightarrow K(\pi_q(BP\langle n \rangle_k), q+1)$$

$$\beta_{#}$$ and $$\gamma_{#}$$ give the split short exact sequence 1.2, 1.5. We know that the $$k$$-invariants in $$H^{q+1}(BP\langle n \rangle_k)^{k+s}, Z(p)$$ are independent and hit $$p$$ torsion generators for $$s < m$$ by statement $$K(n,s), s < m$$ of A and comment 4.4; equivalently, $$(\tau')_{q}$$ is injective:

$$H^q(K(\pi_q(BP\langle n \rangle_k), q)) \rightarrow H^q+1((BP\langle n \rangle_k)^{k+s}) \downarrow^{(\tau')_{q}}$$

$$H^q(K(\pi_q(BP\langle n+1 \rangle_k), q)) \rightarrow H^q+1((BP\langle n+1 \rangle_k)^{k+s}) \downarrow^{(\beta_{#})_{q}}$$

Assume for a moment that $$\gamma_{q-1}$$ pulls these $$k$$-invariants in $$H^{q+1}(BP\langle n \rangle_k)^{k+s}, Z(p)$$ back to independent $$p$$-torsion generating $$k$$-invariants in $$H^{q+1}(BP\langle n+1 \rangle_k)^{k+s}, Z(p)$$, i.e. $$(\tau')_{q} (\gamma_{#})_{*} = (\gamma_{q-1})_{*} (\tau')_{q}$$ is injective in 4.8. Then the first possible dependent $$k$$-invariant is of the type discussed in 3.7. There, it was shown to be an independent $$p$$-torsion generating $$k$$-invariant. Assume for some minimum $$s < m$$ that we have a dependent $$k$$-invariant, or one which is not a $$p$$-torsion generator, equivalently, assume there is an $$x \in \ker(\tau)_{q}$$, therefore $$q > i$$. By what we have assumed about the $$k$$-invariants pulling back, $$x$$ is not in the image of $$(\gamma_{#})_{*}$$. Thus, by the split exactness of homotopy, and therefore the $$(\gamma_{#})_{*}, (\beta_{#})_{*}$$ sequence of 4.8, $$(\beta_{#})_{*}(x) = y \neq 0$$. Now using the result 2.3 about the $$k$$-invariants of loop spaces, $$(s')^{*}\cdot(\tau')_{q} = (\tau')_{q-\tau}^{*} = r$$, $$r = i - k = 2(p^{n+1} - 1)$$. By our minimality assumption on $$s$$, $$(\tau')_{q-\tau}$$ is injective so $$0 \neq (\tau')_{q-\tau}^{*}(y) = (s')^{*}\cdot(\tau')_{q}$$, contradicting $$(\tau)_{q}(x) = 0$. 

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All we need now is to show that $\langle \bar{\tau} \rangle_q \cdot (\gamma_\#)^* = (\gamma^{q-1})^* \cdot (\bar{\tau}^\#)_q$ is injective. We have the maps $(k + s + 1 = q)$

$$(BP_k)^{k+s} \xrightarrow{F^{q-1}} (BP^{n+1})^k \xrightarrow{\gamma^{q-1}} (BP^n)^k$$

If we show that $(F^{q-1})^* \cdot (\gamma^{q-1})^* \cdot (\bar{\tau}^\#)_q$ is injective, we will be through. Using statement $P(n, s)$, $s < m$, from our given in A, we see that this is true if $G^* \cdot (\bar{\tau}^\#)_{k+s+1}$ is injective (as $k = k_{n+1} > k_n$), $G$ the projection:

$$(BP^n)^{k+s} \times \prod_{j > n} (BP^j)^{k+s} \rightarrow (BP^n)^{k+s}$$

This follows trivially from statement $K(n, s)$, $s < m$.

Proof of Statement B. By (1) of statement B and 2.4 on k-invariants of product spaces, all of the k-invariants on the right hand side of $P(n, m)$ are independent and hit p-torsion generators except possibly a zero k-invariant if $m = 2p^l - 3$ which corresponds by construction (3.6) to a dependent k-invariant on the left hand side of $P(n, m)$. Now by $P(n, m)$, $(f^{k+m})^*$ is an isomorphism and so pulls back all of the independent p-torsion generating k-invariants to independent p-torsion generating k-invariants in $(BP^n)^{k+m}$. This determines all of the k-invariants on the left hand side because we know the homotopy is the same on both sides (4.2). So by this, and 3.6 if $m = 2p^l - 3$ $f_\#$ on $\pi_{k+m} + m + 1$ must be an isomorphism. Thus $(f^{k+m+1})_\#$ is an isomorphism on $\pi_\#$ giving us $P(n, m + 1)$.

5. Statement of Results.

In section 4 we proved the main theorem: $k < 2(p^n + \cdots + p + 1)$

$$BP_k \equiv BP^n_k \times \prod_{j > n} BP^j_{k + 2(p^j - 1)}$$

The main theorem of [20] says: The $Z_{(p)}$ (co)homology of the connected part of $BP_k$ has no torsion and is a polynomial algebra for $k$ even and an exterior algebra for $k$ odd. The map above is a map of H-spaces for $k < 2(p^n + \cdots + p + 1)$ so we have the following corollary.

Corollary 5.1. For $k < 2(p^n + \cdots + p + 1)$, the $Z_{(p)}$ (co)homology of the connected part of $BP^n_k$ has no torsion and is a polynomial algebra for $k$ even and an exterior algebra for $k$ odd. For $k = 2(p^n + \cdots + p + 1)$, $H^*(BP^n_k, Z_{(p)})$ has no torsion and is a polynomial algebra. (Note that for $k > 0$ or $k$ odd $< 0$, $BP^n_k$ is connected.)
Note. For \( k = 2(p^n + \cdots + p + 1) \), \( H_*(BP \langle n \rangle_k, \mathbb{Z}(p)) \) is not a polynomial algebra.

At the rationals, the space \( BP \langle n \rangle_k \) is just a product of Eilenberg-MacLane spaces. So, since there is no torsion, the number of generators over \( \mathbb{Z}(p) \) is the same as over \( Q \). As an example, for \( k \) even, \( 0 \leq k \leq 2(p^n + \cdots + p + 1) \), we have

\[
H^*(BP \langle n \rangle_k, \mathbb{Z}(p)) = \mathbb{Z}(p)[s^k \pi^S_s(BP \langle n \rangle)] = \mathbb{Z}(p)[s^k(\mathbb{Z}(p)[x_1, \ldots, x_n])]
\]

For \( k \) even and less than \( 2(p^n + \cdots + p + 1) \), the (co)homology Hopf algebras of 5.1 are bipolynomial, that is, both it and its dual are polynomial algebras. Such Hopf algebras are studied in [13]. There, such a Hopf algebra is shown to be isomorphic to a tensor product of the Hopf algebras \( B(p)[x, 2d] \) studied in [7]. \( B(p)[x, 2d] \), as an algebra, is a polynomial algebra over \( \mathbb{Z}(p) \) on generators \( a_k(x) \) of degree \( 2pkd \). As a Hopf algebra it is isomorphic to its own dual.

Letting \( R(n,k) \) be the set of all \( n \)-tuples of non-negative integers, \( R = (r_1, \ldots, r_n) \) with \( d(R) = 2k + \Sigma 2(p^j - 1)r_j \). \( R \) is called prime if it cannot be written \( R = pR' + (k, 0, \ldots, 0) \) with \( R' \in R(n,k) \). Then, as a further example, we have the following corollary from [13] and the counting done above.

**Corollary 5.2.** For \( 0 \leq k < p^n + \cdots + p + 1 \) as Hopf algebras:

\[
H^*(BP \langle n \rangle_{2k}, \mathbb{Z}(p)) \cong \bigotimes_{R \in R(n,k) \text{ prime}} B(p)[x_R, d(R)]
\]

We now utilize statement \( K(n,s) \); all \( k \)-invariants \( \tau(x) \) in \( H^{k_1+s+2}(BP \langle n \rangle_{k_1+s}, \mathbb{Z}(p)) \) are independent and hit \( p \) torsion generators, \( k_n = 2(p^{n-1} + \cdots + p + 1) + 1 \). This implies that \( BP \langle n \rangle_{k_n} \) cannot be written as a non-trivial product. (2.12)

**Corollary 5.3.** For \( k > 2(p^{n-1} + \cdots + p + 1) \), \( BP \langle n \rangle_k \) is irreducible.

Using the fact that \( k_n + 2(p^j - 1) \geq k_j \) for \( j > n \) we have now completed the proof of the main theorem.

**Theorem 5.4.** For \( k < 2(p^n + \cdots + p + 1) \)

\[
BP_k \cong \bigotimes_{j > n} BP \langle j \rangle_{k+2(p^j - 1)}
\]

and for \( k > 2(p^{n-1} + \cdots + p + 1) \), this decomposition is as irreducibles.

Note. For \( k < 2(p^n + \cdots + p + 1) \) this is as \( H \)-spaces.

Now letting \( k < 2(p^{n-1} + \cdots + p + 1) \) and using two versions of 5.4 we
have
\[ BP_k \cong BP\langle n \rangle_k \times \text{OTHER} \]
and
\[ BP_k \cong BP\langle n - 1 \rangle_k \times BP\langle n \rangle_{k+2(p^n-1)} \times \text{OTHER} \]

From this we get the following corollary.

**Corollary 5.5.** For \( k \leq 2(p^n-1 + \cdots + p+1) \)
\[ BP\langle n \rangle_k \cong BP\langle n - 1 \rangle_k \times BP\langle n \rangle_{k+2(p^n-1)} \]

*Note.* For \( k \leq 2(p^n-1 + \cdots + p+1) \) this is as \( H \)-spaces.

This gives us the point where the fibration 1.4 becomes trivial. Again, using
\[ BP_k \cong BP\langle n \rangle_k \times \text{OTHER} \] for \( k \leq 2(p^n + \cdots + p+1) \) and the fact that for finite complexes \( BP\langle n \rangle^k(X) = 0 \) for high \( k \) we get 5.6.

**Corollary 5.6.** (i) \( BP^k(X) \to BP\langle n \rangle^k(X) \) is onto for \( k \leq 2(p^n + \cdots + p+1) \). (ii) \( BP^*(X) \to BP\langle n \rangle^*(X) \) is onto in all but a finite number of dimensions.

We now apply 5.6 to prove Quillen’s Theorem. The problem was first studied in [6].

**Theorem 5.7** (Quillen). Let \( X \) be a finite CW complex, then \( BP^*(X) \) is generated as a \( BP^*(S^0) \) module by elements of non-negative degree.

*Proof.* If \( u \in BP^k(X) \) and \( k < 0 \), we will show \( u \) is a finite sum \( \sum_i x_i u_i = u \), \( u_i \in BP^{k+2(p^n-1)}(X) \) and \( x_i \in BP^*(S^0) = Z_p[x_1, \ldots, x_i, \ldots] \) of degree \( -2(p^n-1) \). By downward induction on the degree of \( u \) we will be done.

Consider the maps
\[ BP^*(X) \xrightarrow{g_n} BP\langle n \rangle^*(X) \xrightarrow{f_n} BP\langle n-1 \rangle^*(X) \]
Find \( n \) such that \( g_n(u) \neq 0 \) but \( f_n \cdot g_n(u) = g_{n-1}(u) = 0 \). Such an \( n \) exists because \( n = 0 \) gives \( g_0(u) \in H^k(X, Z_p) = 0 \) as \( k < 0 \), and for \( n \) high enough \( BP^k(X) \cong BP\langle n \rangle^k(X) \), by the finiteness of \( X \).

Dual to 1.1 we have an exact sequence and commuting diagram:

\[ \begin{array}{c}
BP^{k+2(p^n-1)}(X) \xrightarrow{x_n} BP^k(X) \\
\downarrow g_n \quad \downarrow g_n
\end{array} \]
\[ \begin{array}{c}
BP\langle n \rangle^{k+2(p^n-1)}(X) \xrightarrow{x_n} BP\langle n \rangle^k(X) \xrightarrow{f_n} BP\langle n-1 \rangle^k(X)
\end{array} \]
\[ u' \xrightarrow{g_n(u)} 0 \]
As $f_n(g_n(u))=0$ there exists $u'$ with $x_nu'=g_n(u)$ by exactness. But now, by 5.6 and $2(p^n+\cdots+p+1)>k+2(p^n-1)$ for $k<0$ we have that $g_n$ is onto in dimension $k+2(p^n-1)$ and so pick $u_n\in BP^{k+2(p^n-1)}(X)$ with $g_n(u_n)=u'$. Then by commutativity, $g_n(x_nu_n)=g_n(u)$. Now continue this process using $u-x_nu_n$. By the finiteness of $X$, $BP^{k+2(p^n-1)}(X)$ will be zero for large $j$ and we will get our finite sum $u=\sum_{i>0}x_iu_i$ and be done.

The spaces $BP_n$ are most useful in the range $2(p^n-1+\cdots+p+1)<k<2(p^n+\cdots+p+1)$ where they are both irreducible and torsion free. In the next section, we will identify these with spaces that have perhaps a more tangible description.


All modules will be over $Z(p)$, and, until further notice, all coefficients will be $Z(p)$. In this section we will study torsion free $H$-spaces. Our immediate goal is to construct and study the following spaces.

**Proposition 6.1.** There exists an irreducible $k-1$ connected $H$-space $Y_k$ which has $H^*(Y_k)$ and $\pi_*(Y_k)$ both free over $Z(p)$ and such that each stage of the Postnikov system is irreducible.

**Proof.** We will build up a Postnikov system for $Y_k$ and use 2.7. We drop the subscript $k$. Clearly we must start the Postnikov system with $Y^k=K(Z(p),k)$. We will now just build up a Postnikov system by killing off the torsion in cohomology as efficiently as possible. $\pi_*(Y^k)$ is free over $Z(p)$ and $H^j(Y^k)$ has no torsion for $j \leq k+1$. $Y^k$ is an $H$-space. Assume we have constructed the $s-1$ stage, $Y^{s-1}$ for $s>k$ such that $\pi_*(Y^{s-1})$ is free and $H^j(Y^{s-1})$ has no torsion for $j \leq s$. Assume also that $Y^{s-1}$ has an $H$-space structure. $H^{s+1}(Y^{s-1})\cong F\oplus T$ where $F$ is the free part and $T$ is the torsion part. It is finitely generated so it is isomorphic to $(Z(p))^{n_0}\oplus\bigoplus_{i>0}(Z(p))^{n_i}$, where $(G)^n=G\oplus\cdots\oplus G$ $n$ times. Using the torsion generators, this isomorphism determines a map:

$$Y^{s-1}\longrightarrow \prod_{n=\sum_{i>0}n_i\text{ times}}K(Z(p),s+1)=K(F_n,s+1), \quad F_n=(Z(p))^{n_i}$$

Let this map be the $s$ $k$-invariant, $k_s$. This constructs the space $Y^s$ as the induced fibration. $k_s$ is torsion and so it is primitive because there is no torsion in lower dimensions, therefore by 2.5, $Y^s$ is an $H$-space. Recall the $Z(p)$
sequence 2.9

\[
\begin{array}{c}
0 \rightarrow H^*(Y^{s-1}) \xrightarrow{(g_\ast)^*} H^*(Y^s) \rightarrow H^*(K(F_n, s)) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Choose $V^s \subset H^s(X^s) = H^s(X)$ such that $i^*: V^s \to \ker \gamma$. Let $r_s$ be the rank of $V^s$. This determines a map $f'_s: X \to K((Z_p)^s, s)$. $H^*(X, Z(p))$ has no $p$ torsion so $f'_s$ lifts first into the product of $r_s$ copies of $K(Z(p), s)$ and then into the product of $r_s$ copies of $Y_s$, denoted $rY_s$. (It lifts by 2.6 because $Y_s$ has $p$-torsion $k$-invariants and free homotopy.) So we have $f_s: X \to rY_s$ such that the image of $(f_s)^*$ in dimension $s$ is $V^s$. Let $f = \prod_s f_s: X \to \prod_s rY_s = Y$.

**Claim.** $f$ is a homotopy equivalence.

**Proof.** By induction on the Postnikov system assume $f^s: X^s \to Y^s = \prod_{k \leq s} (rY_k)^s$ is a homotopy equivalence. $(X^1 = Y^1 = pt)$ Let $f_#$ be the induced map $K(\pi_{s+1}(X), s+1) \to K(\pi_{s+1}(Y), s+1)$. $f^s$ is a homotopy equivalence, if $f_#$ is too, then $f^{s+1}_#: \pi_s(X^{s+1}) \to \pi_s(Y^{s+1})$ is an isomorphism and so $f^{s+1}$ is a homotopy equivalence. $f_#$ is a homotopy equivalence iff $(f_#)^*: H^{s+1}(K(\pi_{s+1}(X), s+1)) \to H^{s+1}(K(\pi_{s+1}(Y), s+1))$ is an isomorphism. Now

$$K(\pi_{s+1}(Y), s+1) = K\left(\prod_{k \leq s} Y_k, s+1\right) \times K((Z(p))^{s+1}, s+1) = K \times K'$$

and $H^{s+1}(K \times K') = H^{s+1}(K) \oplus H^{s+1}(K')$. $\ker \gamma = H^{s+1}(K')$ by the construction of the $Y_k$, i.e., all $k$-invariants are independent and hit $p$-torsion generators. Using the naturality of 2.9 we have:

$$\begin{align*}
0 & \longrightarrow V^{s+1} \longrightarrow H^{s+1}(X^{s+1}) \quad (f^s)^* \quad \longrightarrow H^{s+1}(K(\pi_{s+1}(X), s+1)) \quad \longrightarrow H^{s+2}(X^s) \\
\uparrow \cong & \quad \uparrow i^*_X \quad \uparrow (f^s)^* \quad \uparrow \gamma \quad \uparrow \gamma \quad \uparrow \gamma \quad \longrightarrow H^{s+1}(Y^{s+1}) \quad \longrightarrow H^{s+1}(K(\pi_{s+1}(Y), s+1)) \quad \longrightarrow H^{s+2}(Y^s) \quad \longrightarrow (6.4)
\end{align*}$$

Now $(i_Y)^*: H^{s+1}(rY_{s+1}) \to \ker \gamma = H^{s+1}(K')$ and by construction of $f_{s+1}$, $f^* = (f^{s+1})^*: H^{s+1}(rY_{s+1}) \to V^{s+1}$. By commutativity, $(f_#)^*: \ker \gamma \to \ker \gamma_X$. $\gamma|H^{s+1}(K)$ is injective and by our construction of the $Y_k$ it hits every possible element in cohomology, i.e., all that reduce from torsion elements in $Z(p)$ cohomology. $f^s$ is a homotopy equivalence by induction so by commutativity of 6.4 $\gamma_X$ also hits all possible elements and we have isomorphism on the ends of diagram 6.5 giving us the desired isomorphism by the five lemma.

$$\begin{align*}
0 & \quad \longrightarrow V^{s+1} \quad \longrightarrow H^{s+1}(K(\pi_{s+1}(X), s+1)) \quad \longrightarrow \text{image } \gamma_X \longrightarrow 0 \\
\uparrow \cong & \quad \quad \uparrow (f_#)^* \quad \quad \uparrow \gamma \quad \quad \uparrow \gamma \quad \quad \longrightarrow H^{s+1}(K') \oplus H^{s+1}(K) \quad \longrightarrow \text{image } \gamma_Y \longrightarrow 0 \quad (6.5)
\end{align*}$$
Corollary 6.6. $Y_k$ as in 6.1 is unique up to homotopy type.

Corollary 6.7. Any map $Y_k \to Y_k$ which induces an isomorphism on $\pi_k(Y_k) = Z_{(p)}$ is a homotopy equivalence.

Corollary 6.8. For $2(p^{n-1} + \cdots + p + 1) < k < 2(p^n + \cdots + p + 1)$, $BP\langle n \rangle_k \simeq Y_k$.

Proof. $\pi_*(BP\langle n \rangle_k)$ is free by construction. (1.5) $H^*(BP\langle n \rangle_k)$ has no torsion for $k < 2(p^n + \cdots + p + 1)$ by 5.1. For $k > 2(p^{n-1} + \cdots + p + 1)$, $BP\langle n \rangle_k$ is irreducible by 5.3. Now, for $k$ in this range just apply 6.2.

Corollary 6.9. For $k > 2(p^{n-1} + \cdots + p + 1)$, any map $BP\langle n \rangle_k \to BP\langle n \rangle_k$ which induces an isomorphism on $\pi_k$ is a homotopy equivalence.

Note that $Y_\infty \simeq BP$ and we get an unpublished result of F. P. Peterson.

Corollary 6.10 (Peterson). Given a spectrum $X$ with $H^*(X,Z_{(p)})$ and $\pi_*(X)$ bounded below, locally finitely generated, and free over $Z_{(p)}$, then

$$X \simeq V_1 S^k BP$$

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REFERENCES.


