ON NOVIKOV'S EXT^1 MODULO AN INVARIANT PRIME IDEAL

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This note is a statement of some results on $Ext^{1,*}_{BP_*BP}(BP_*, BP_*/I_n)$ which we talked about informally at the summer 1974 homotopy-theory conference at Northwestern University. Proofs will appear elsewhere. For details on the Brown-Peterson spectrum BP and on BP^*BP and BP_*BP, we refer the reader to [2, 11, 1].

We shall use the generators v_i of Hazewinkel [3], so that

$$BP_* \simeq Z_{(p)}[v_1, v_2, \ldots]$$

with $|v_n| = 2p^n-2$, and $BP_* \simeq BP^{-*}$. The ideals

$$I_n = (p, v_1, \ldots, v_{n-1}) \quad 0 \leq n \leq \infty$$

are the prime ideals of BP_* invariant under the coaction of BP_*BP (or the action of BP^*BP); see [5, 9, 4]. We point out that

$$Ext^{**}_{BP^*BP}(BP^*, BP^*/I_n) \simeq Ext^{**}_{BP_*BP}(BP_*, BP_*/I_n),$$

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and henceforth denote this algebra by

$$\mathrm{Ext}^{**}(BP_*, BP_*/I_n).$$

Multiplication by v_n on BP_*/I_n is a BP_*BP-comodule map. In fact, we have

<u>Theorem</u> (Landweber [14]; see also Johnson-Wilson [4]). For $0 < n < \infty$,

$$\mathrm{Ext}^{0,*}(BP_*, BP_*/I_n) \simeq \mathbb{F}_p[v_n].$$

Thus $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ splits up as an $\mathbb{F}_p[v_n]$-module into a direct sum of v_n-torsion and v_n-torsion-free submodules. For p odd, we describe the v_n-torsion summand completely, and exhibit all but one generator for the v_n-torsion-free summand.

The short exact sequence of comodules (where $v_0 = p$)

$$0 \longrightarrow BP_*/I_n \xrightarrow{\ v_n\ } BP_*/I_n \longrightarrow BP_*/I_{n+1} \longrightarrow 0$$

gives rise to the "Bockstein" exact couple

$$\begin{array}{ccc} \mathrm{Ext}^{**}(BP_*, BP_*/I_n) & \xrightarrow{\ v_n\ } & \mathrm{Ext}^{**}(BP_*, BP_*/I_n) \\ {\scriptstyle \delta_n}\nwarrow & & \swarrow{\scriptstyle \rho_n} \\ & \mathrm{Ext}^{**}(BP_*, BP_*/I_{n+1}) & \end{array}$$

in which δ_n has bidegree $(1,2-2p^n)$.

Henceforth let p be an odd prime. Recall [1] that $BP_*BP \simeq BP_*[t_1,t_2,\ldots]$, $|t_n| = 2p^n-2$. In the cobar construction for BP_*BP ([7]) with coefficients in BP_*/I_n, $n>0$, $[t_1^{p^i}]$ is cycle representing a nonzero class

$$h_i \in \mathrm{Ext}^{1,p^iq}(BP_*,BP_*/I_n),$$

$q = 2p-2$. Clearly h_i is taken to h_i by the reduction ρ_n.

Note that

$$\mathrm{Ext}^{**}(BP_*,BP_*/I_\infty) \simeq \mathrm{Ext}^{**}_{P_*}(\mathbb{F}_p,\mathbb{F}_p)$$

where P_* is the Hopf algebra of Steenrod reduced powers. Thus $\mathrm{Ext}^{1,*}(BP_*,BP_*/I_\infty)$ is additively generated by $\{h_i : i \geq 0\}$ [6]. (At the other extreme recall that Novikov [10] has computed $\mathrm{Ext}^{1,*}(BP_*,BP_*/I_0)$.)

<u>Theorem</u> A. Let p be odd and $0<n<\infty$. All relations in the $\mathbb{F}_p[v_n]$-submodule of $\mathrm{Ext}^{1,*}(BP_*,BP_*/I_n)$ generated by $\{h_i : i \geq 0\}$ are consequences of

$$v_n^{p^s} h_{s+n} = v_n^{p^{s+1}} h_s \qquad s \geq 0.$$

<u>Corollary</u> A'. The h_i for $0 \leq i < n$ generate distinct free $\mathbb{F}_p[v_n]$-module summands.

The next theorem describes the v_n-torsion submodule of $\mathrm{Ext}^{1,*}(BP_*,BP_*/I_n)$, $0<n<\infty$. For $r>0$, write $r = ap^s$ with

$(a,p) = 1$, and if $s \neq 0$ write $s = kn + i + 1$ with $0 \leq i < n$.

Let

$$q(r) = q_n(r) = \begin{cases} p^s & \text{if } a = 1 \\ p^s + (p-1)\sum_{\ell=0}^{k-1} p^{\ell n+i} & \text{if } a \neq 1 \end{cases}$$

In particular, for $n = 1$ with $s > 1$ and $a \neq 1$, $q(ap^s) = p^s + p^{s-1} - 1$.

<u>Theorem</u> B. Let p be odd and $0 < n < \infty$. The v_n-torsion submodule of $\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n)$ is a sum of cyclic $\mathbb{F}_p[v_n]$-modules on generators

$$c_n(r) \in \mathrm{Ext}^{1,2r(p^{n+1}-1)-2q(r)(p^n-1)}(BP_*, BP_*/I_n)$$

satisfying, for a such that $(a,p) = 1$ and $a \neq 1$:

(i) $v_n^{q(r)} c_n(r) = 0$

$$v_n^{q(r)-1} c_n(r) = \delta_n(v_{n+1}^r) \neq 0$$

(ii) $h_{s+n} = c_n(p^s) + v_n^{p^s(p-1)} h_s \quad s \geq 0$

(iii) $\rho_n(c_n(p^s)) = h_{s+n}$

$$\rho_n(c_n(ap^0)) = a v_{n+1}^{a-1} h_n$$

$$\rho_n(c_n(ap^s)) = \begin{cases} 2a v_2^{ap^s - p^{s-1}} h_0 & \text{if } n = 1 \text{ and } s > 1. \\ a v_{n+1}^{ap^s - p^{s-1}} h_i & \text{otherwise.} \end{cases}$$

Most of our understanding of the v_n-torsion-free part of $Ext^{1,*}(BP_*, BP_*/I_n)$ derives from the following theorem of Morava.

<u>Theorem</u> (Morava [8]). Let p be odd. The rank of $Ext^{1,*}(BP_*, BP_*/I_n)$ over $\mathbb{F}_p[v_n]$ is 1 for $n = 1$, and $n+1$ for $1 < n < \infty$.

Corollary A' gives us all but one generator of $Ext^{1,*}(BP_*, BP_*/I_n)$ mod v_n-torsion if $n > 1$. For the last generator we can only offer:

<u>Conjecture</u>. For p odd and $1 < n < \infty$, there is an element $w_n \in Ext^{1,*}(BP_*, BP_*/I_n)$ generating a free $\mathbb{F}_p[v_n]$-module summand and reducing to

$$\rho_n(w_n) = v_{n+1}^{1+p+\ldots+p^{n-2}} h_{n-1}.$$

Our principal evidence for this conjecture is its truth for $n = 2$ and 3.

These results have applications in stable homotopy. It is immediate from Theorem B that $\delta_0\delta_1(v_2^t) \neq 0$ in $Ext^{2,*}(BP_*, BP_*)$ for $t > 0$. This implies the theorem of L. Smith [12] that $\beta_t \neq 0$ in π_*^S for $t > 0$.

Recall [10] that the image of

$$\rho_0 : Ext^{1,*}(BP_*, BP_*) \longrightarrow Ext^{1,*}(BP_*, BP_*/(p))$$

is generated by $\left\{v_1^k h_0 : k \geq 0\right\}$. Since $Ext^{2,*}(BP_*, BP_*)$ is p-torsion, the exact sequence

$$\mathrm{Ext}^{1,*}(BP_*,BP_*)\xrightarrow{\rho_0}\mathrm{Ext}^{1,*}(BP_*,BP_*/(p))\xrightarrow{\delta_0}\mathrm{Ext}^{2,*}(BP_*,BP_*)\xrightarrow{p}\mathrm{Ext}^{2,*}(BP_*,BP_*)$$

allows us to compute the kernel of multiplication by p in $\mathrm{Ext}^{2,*}(BP_*,BP_*)$. This gives a complete list of cyclic $Z_{(p)}$-module summands, but no information on their orders. Using this list it is easy to see that $\delta_0\delta_1\delta_2(v_3)\neq 0$ in $\mathrm{Ext}^{3,*}(BP_*,BP_*)$. This implies the result of E. Thomas and R.S. Zahler [13] that $\gamma_1\neq 0$ in π_*^S.

In a following note with D.C. Johnson and R.S. Zahler we describe this technique in more detail and use it to show the nontriviality of a sporadic but infinite collection of γ_t's.

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