ON NOVIKOV'S $\text{Ext}^1$ MODULO AN INVARIANT PRIME IDEAL

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This note is a statement of some results on

$\text{Ext}^1_{BP^*BP}(BP^*, BP^*/I^n)$

which we talked about informally at the summer 1974 homotopy-theory conference at Northwestern University. Proofs will appear elsewhere. For details on the Brown-Peterson spectrum $BP^*$ and on $BP^* BP$ and $BP^* BP^*$, we refer the reader to [2, 11, 1].

We shall use the generators $v_i$ of Hazewinkel [3], so that

$$BP^* \simeq \mathbb{Z}(p)[v_1, v_2, \ldots]$$

with $|v_n| = 2p^{n-2}$, and $BP^* \simeq BP^*$. The ideals

$$I_n = (p, v_1, \ldots, v_{n-1}) \quad 0 \leq n \leq \infty$$

are the prime ideals of $BP^*$ invariant under the coaction of

$BP^* BP$ (or the action of $BP^* BP$); see [5, 9, 4]. We point out that

$$\text{Ext}^{**}_{BP^* BP}(BP^*, BP^*/I^n) \simeq \text{Ext}^{**}_{BP^* BP}(BP^*, BP^*/I^n)$$

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and henceforth denote this algebra by

$$\text{Ext}^{**}(BP_*, BP_*/I_n).$$

Multiplication by $v_n$ on $BP_*/I_n$ is a $BP_*/BP$-comodule map.

In fact, we have

**Theorem** (Landweber [14]; see also Johnson-Wilson [4]). For $0 < n < \infty$,

$$\text{Ext}^{O,*}(BP_*, BP_*/I_n) \simeq \mathbb{F}_p[v_n].$$

Thus $\text{Ext}^{1,*}(BP_*, BP_*/I_n)$ splits up as an $\mathbb{F}_p[v_n]$-module into a direct sum of $v_n$-torsion and $v_n$-torsion-free submodules.

For $p$ odd, we describe the $v_n$-torsion summand completely, and exhibit all but one generator for the $v_n$-torsion-free summand.

The short exact sequence of comodules (where $v_0 = p$)

$$0 \rightarrow BP_*/I_n \stackrel{v_n}{\rightarrow} BP_*/I_n \rightarrow BP_*/I_{n+1} \rightarrow 0$$

gives rise to the "Bockstein" exact couple

$$\begin{array}{ccc}
\text{Ext}^{**}(BP_*, BP_*/I_n) & \stackrel{v_n}{\rightarrow} & \text{Ext}^{**}(BP_*, BP_*/I_n) \\
\delta_n & \downarrow & \rho_n \\
\text{Ext}^{**}(BP_*, BP_*/I_{n+1}) & & \\
\end{array}$$
in which $\delta_n$ has bidegree $(1,2-2p^n)$.

Henceforth let $p$ be an odd prime. Recall [1] that $BP_*BP \cong BP_*[t_1, t_2, \ldots], |t_n| = 2p^n - 2$. In the cobar construction for $BP_*BP$ ([7]) with coefficients in $BP_*/I_n, n > 0,$ $[t_1^p]$ is a cycle representing a nonzero class

$$h_i \in \text{Ext}^1_{BP_*/BP_*/I_n}.$$ q = 2p-2. Clearly $h_i$ is taken to $h_i$ by the reduction $p_n$.

Note that

$$\text{Ext}^{**}(BP_*/BP_*/I_n) \cong \text{Ext}^{**}(IF_*/IF_*)$$

where $P_*$ is the Hopf algebra of Steenrod reduced powers. Thus $\text{Ext}^1_{BP_*/BP_*/I_n}$ is additively generated by $\{h_i : i \geq 0\}$ [6].

(At the other extreme recall that Novikov [10] has computed $\text{Ext}^1_{BP_*/BP_*/I_0}$.)

Theorem A. Let $p$ be odd and $0 < n < \infty$. All relations in the $IF_*/v_n^s$-submodule of $\text{Ext}^1_{BP_*/BP_*/I_n}$ generated by $\{h_i : i \geq 0\}$ are consequences of

$$v_n^p h_{s+n} = v_n^{p s+1} h_s \quad s \geq 0.$$

Corollary A'. The $h_i$ for $0 \leq i < n$ generate distinct free $IF_*/v_n^s$-module summands.

The next theorem describes the $v_n$-torsion submodule of $\text{Ext}^1_{BP_*/BP_*/I_n}, 0 < n < \infty$. For $r > 0$, write $r = ap^s$ with
(a, p) = 1, and if \( s \neq 0 \) write \( s = kn + i + 1 \) with \( 0 \leq i < n \).

Let

\[
q(r) = q_n(r) = \begin{cases} 
  s & \text{if } a = 1 \\
  p^s + (p-1) \sum_{k=0}^{k-1} p^{ln+i} & \text{if } a \neq 1
\end{cases}
\]

In particular, for \( n = 1 \) with \( s > 1 \) and \( a \neq 1 \), \( q(ap^s) = p^s + p^{s-1} \).

**Theorem B.** Let \( p \) be odd and \( 0 < n < \infty \). The \( v_n \)-torsion submodule of \( \text{Ext}_{BP, BP}^1(BP_n, BP_n/I_n) \) is a sum of cyclic \( \mathbb{F}_p[v_n] \)-modules on generators

\[
c_n(r) \in \text{Ext}_{BP, BP}^1(BP_n, BP_n/I_n)
\]

satisfying, for \( a \) such that \( (a, p) = 1 \) and \( a \neq 1 \):

(i) \( v_n^{q(r)} c_n(r) = 0 \)

(ii) \( v_n^{q(r)-1} c_n(r) = \delta_{n,v_n^{n+1}} \neq 0 \)

(iii) \( h_{s+n} = c_n(p^s) + v_n^{p^s(p-1)} h_s \quad s \geq 0 \)

(iv) \( \rho_n(c_n(p^s)) = h_{s+n} \)

(v) \( \rho_n(c_n(ap^0)) = av_{n+1}^{a-1} h_0 \)

\( \rho_n(c_n(ap^s)) = \begin{cases} 
  2av_{n+1}^{ap^s-p^s-1} h_0 & \text{if } n = 1 \text{ and } s > 1. \\
  av_{n+1}^{ap^s-p^s-1} h_i & \text{otherwise.}
\end{cases} \)
Most of our understanding of the $v_n$-torsion-free part of $\text{Ext}^{1,*}(BP_*^*,BP_*^*/I_n^*)$ derives from the following theorem of Morava.

**Theorem (Morava [8]).** Let $p$ be odd. The rank of $\text{Ext}^{1,*}(BP_*^*,BP_*^*/I_n^*)$ over $\mathbb{F}_p[v_n]$ is $1$ for $n = 1$, and $n+1$ for $1 < n < \infty$.

Corollary A' gives us all but one generator of $\text{Ext}^{1,*}(BP_*^*,BP_*^*/I_n^*) \mod v_n$-torsion if $n > 1$. For the last generator we can only offer:

**Conjecture.** For $p$ odd and $1 < n < \infty$, there is an element $w_n \in \text{Ext}^{1,*}(BP_*^*,BP_*^*/I_n^*)$ generating a free $\mathbb{F}_p[v_n]$-module summand and reducing to

$$\rho_n(w_n) = v_0^{1+p+\ldots+p^{n-2}} h_{n-1}.$$  

Our principal evidence for this conjecture is its truth for $n = 2$ and $3$.

These results have applications in stable homotopy. It is immediate from Theorem B that $\delta_1(v_2^{t}) \neq 0$ in $\text{Ext}^{2,*}(BP_*^*,BP_*^*)$ for $t > 0$. This implies the theorem of L. Smith [12] that $\beta_t \neq 0$ in $\pi_*^S$ for $t > 0$.

Recall [10] that the image of

$$\rho_0 : \text{Ext}^{1,*}(BP_*^*,BP_*^*) \rightarrow \text{Ext}^{1,*}(BP_*^*,BP_*^*/(p))$$

is generated by $\{v_0^k h_0 : k \geq 0\}$. Since $\text{Ext}^{2,*}(BP_*^*,BP_*^*)$ is
p-torsion, the exact sequence

$$\text{Ext}^1_{(BP_*,BP_*)} \xrightarrow{\partial_0} \text{Ext}^1_{(BP_*,BP_*/(p))} \xrightarrow{\delta_0} \text{Ext}^2_{(BP_*,BP_*)} \xrightarrow{\partial} \text{Ext}^2_{(BP_*,BP_*)}$$

allows us to compute the kernel of multiplication by $p$ in $\text{Ext}^2_{(BP_*,BP_*)}$. This gives a complete list of cyclic $Z(p)$-module summands, but no information on their orders. Using this list it is easy to see that $\delta_0 \delta_1 \delta_2(v_3) \neq 0$ in $\text{Ext}^3_{(BP_*,BP_*)}$. This implies the result of E. Thomas and R.S. Zahler [13] that $\gamma_1 \neq 0$ in $\pi^8_\ast$.

In a following note with D.C. Johnson and R.S. Zahler we describe this technique in more detail and use it to show the nontriviality of a sporadic but infinite collection of $\gamma_t$'s.

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This announcement will appear in the proceedings for the
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