BOUNDARY HOMOMORPHISMS IN THE GENERALIZED ADAMS SPECTRAL SEQUENCE AND THE NONTRIVIALITY OF INFINITELY MANY γ_t IN STABLE HOMOTOPY

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We apply the computation announced in [8] to prove the following result on the nontriviality of an infinite subset of the family $\{\gamma_t : t > 0\}$ in the stable homotopy of the sphere.

<u>Theorem</u>. $\gamma_t \in \pi^S_{2(p^3-1)t-2(p^2-1)-2(p-1)-3}(S^0)$ is essential if $t = rp^s$, $r = 2,\ldots,p-1$, $s > 0$.

The elements γ_t have been detected for $t = ap+b$, $0 \leq a < b \leq p-1$, by E. Thomas and R.S. Zahler [12,13]. Several programs for detecting the whole gamma family are currently under way, but as far as we know, none has yet succeeded.

Our approach is to reduce the theorem to an algebraic question in E_2 of the Adams spectral sequence for BP homology and then appeal to [8] and arithmetic to deduce the result. The arithmetic actually shows $\gamma_t \neq 0$ for other values of t in a set of density zero.

Our methods allow a systematic detection of elements in infinite families in stable homotopy. We illustrate this by proving that all the elements in the alpha and beta families are nontrivial, assuming only the existence of the self-maps required for their construction. The same technique could be used to detect the known members of the epsilon family, again assuming their construction.

The link between algebra and homotopy theory is provided in the first section by a folk theorem relating algebraic and geometric connecting homomorphisms. The second section defines the stable homotopy elements of interest to us and uses [8] to detect many of them.

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1. Algebraic Boundaries and Geometric Boundaries.

Our goal in this section is to prove a general result (1.7) relating connecting homomorphisms in E_2 and in the abutment of generalized homology Adams spectral sequences.

We recall the construction [3, III, §15] of an Adams spectral sequence based on a homotopy-associative ring spectrum E with unit $\eta : S^0 \longrightarrow E$. Let $F^0 = S^0$ and for $s \geq 0$ let F^{s+1} complete the cofibration sequence

(1.1) $$F^s \xrightarrow{j_s} E \wedge F^s \xrightarrow{k_s} F^{s+1} \xrightarrow{i_s} F^s$$

in which k_s has degree -1 and

$$j_s = \eta \wedge F^s : F^s \simeq S^0 \wedge F^s \longrightarrow E \wedge F^s.$$

The sequences (1.1) splice together to form an Adams resolution for S^0. When smashed with a connective spectrum X, they form an Adams resolution for X. Note that

(1.2) $$E_*(i_s \wedge X) = 0.$$

If we apply $\pi_*^S(\ )$, we obtain an exact couple whose associated spectral sequence is the E-homology Adams spectral sequence $E_r^{**}(X)$.

Define a filtration of $\pi_*^S(X)$ by

(1.3) $$F^s \pi_*^S(X) = \text{image}\left\{\pi_*^S(F^s \wedge X) \longrightarrow \pi_*^S(X)\right\}.$$

Lift $x \in F^s \pi_*^S(X)$ to $y \in \pi_*^S(F^s \wedge X)$. Then $(j_s \wedge X)y \in \pi_*^S(E \wedge F^s \wedge X) = E_1^{s,*}(X)$ is a permanent cycle and projects to an element of $E_\infty^{s,*}(X)$ which depends only on x modulo $F^{s+1}\pi_*^S(X)$. Thus we have a homomorphism of bigraded modules

(1.4) $$E_0^* \pi_*^S(X) \longrightarrow E_\infty^{**}(X).$$

Now suppose $E_* = E_*(S^0)$ is commutative and $E_*(E)$ is flat over E_*. The left unit $\eta_L : E_* \longrightarrow E_*(E)$ is split by the multiplication map so the cokernel, $E_*(F^1)$ by (1.2), is also flat. Then $E_*(F^1) \otimes_{E_*} E_*(-)$ gives a homology theory naturally equivalent to $E_*(F^1 \wedge -)$. Using the observation that $F^t \simeq F^1 \wedge F^{t-1}$, we prove inductively that:

$$E_*(F^t \wedge X) \cong E_*(F^t) \otimes_{E_*} E_*(X) \tag{1.5}$$

for any connective spectrum X. Then [2,3]

$$E_2^{**}(X) \cong \mathrm{Ext}^{**}_{E_*(E)}(E_*, E_*(X)).$$

This Ext is an Ext of comodules over the "coalgebra" $E_*(E)$; it is computed using extended $E_*(E)$ comodules as injectives.

Definition 1.6. The class $\bar{x} \in \mathrm{Ext}^{t,*}_{E_*(E)}(E_*, E_*(X))$ is said to converge to $x \in \pi_*^S(X)$ provided that

(i) $\bar{x}$ is a permanent cycle representing the class $\{\bar{x}\} \in E_\infty^{t,*}(X)$;

(ii) $x \in F^t \pi_*^S(X)$; and

(iii) The homomorphism (1.4) sends the coset $x + F^{t+1}\pi_*^S(X)$ to $\{\bar{x}\}$.

We define a map $f : X \longrightarrow Y$ to be E-proper provided that $E_*(f) = 0$. (This terminology was suggested by Larry Smith.) If in the cofibration sequence

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} SW$$

the map h is E-proper, we obtain a short exact sequence

$$0 \longrightarrow E_*(W) \xrightarrow{E_*(f)} E_*(X) \xrightarrow{E_*(G)} E_*(Y) \longrightarrow 0.$$

In turn, this induces a long exact sequence

$$\begin{array}{ccc} \mathrm{Ext}^{**}_{E_*(E)}(E_*,E_*(W)) & \longrightarrow & \mathrm{Ext}^{**}_{E_*E}(E_*,E_*(X)) \\ & \nwarrow^{\delta} \quad \swarrow & \\ & \mathrm{Ext}^{**}_{E_*(E)}(E_*,E_*(Y)) & \end{array}$$

where the connecting homomorphism δ is as in [4, p. 55] and has bidegree $(1,0)$.

<u>Theorem</u> 1.7. (<u>Geometric Boundary Theorem</u>) Let E be a homotopy associative ring spectrum with unit such that E_* is commutative and $E_*(E)$ is flat over E_* is commutative and $E_*(E)$ is flat over E_*. Let $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} SW$ be a cofibre sequence of finite spectra with h an E-proper map. If $\bar{x} \in \mathrm{Ext}^{t,*}_{E_*(E)}(E_*,E_*(Y))$ converges to $x \in \pi^S_*(Y)$, then $\delta(\bar{x})$ converges to $h_*(x) \in \pi^S_*(W)$.

<u>Proof</u>. Smash the Adams resolution for the sphere with the cofibration sequence $W \to X \to Y$. Part of the resulting diagram

is displayed in (1.8). Let $y \in \pi_*^S(F^t \wedge Y)$ be such that $(j_t \wedge Y)y$ represents $\bar{x}$ in $E_2^{t,*}(Y)$ and

$$(1.9) \qquad (i_0 \ldots i_{t-1} \wedge Y)y = x.$$

By (1.5), the $E \wedge F^t \wedge -$ row in (1.8) is short in homotopy, so $0 = (E \wedge F^t \wedge h)(j_t \wedge Y)y = (j_t \wedge W)(F^t \wedge h)y$ and there exists $y_1 \in \pi_*^S(F^{t+1} \wedge W)$ such that

$$(1.10) \qquad (i_t \wedge W)y_1 = (F^t \wedge h)y.$$

We come now to the main geometric step.

<u>Claim</u> 1.11. There is an element $y_2 \in \pi_*^S(E \wedge F^t \wedge X)$ such that

$$(j_t \wedge Y)y = (E \wedge F^t \wedge g)y_2,$$

$$(k_t \wedge X)y_2 = (F^{t+1} \wedge f)y_1.$$

To see this, pass to the Spanier-Whitehead dual cofibration sequence $DW \leftarrow DX \leftarrow DY$. Take maps $y^\#$ and $y_1^\#$ dual to y and y_1. We have

$$\begin{array}{ccccc} DX & \leftarrow & DY & \leftarrow & S^{-1}DW \\ \Big\downarrow{\scriptstyle y_2^\#} & & \Big\downarrow{\scriptstyle y^\#} & & \Big\downarrow{\scriptstyle S^{-1}y_1^\#} \\ E \wedge F^t & \leftarrow & F^t & \leftarrow & F^{t+1}. \end{array}$$

Let $y_2^\#$ complete the map of cofibrations (see [16], p. 170).

Then the map $y_2 \in \pi_*^S(E \wedge F^t \wedge Y)$ dual to $y_2^\#$ satisfies the conditions of the claim.

Now it is easy, using the definition of the connecting homomorphism δ, to chase (1.8) and see that

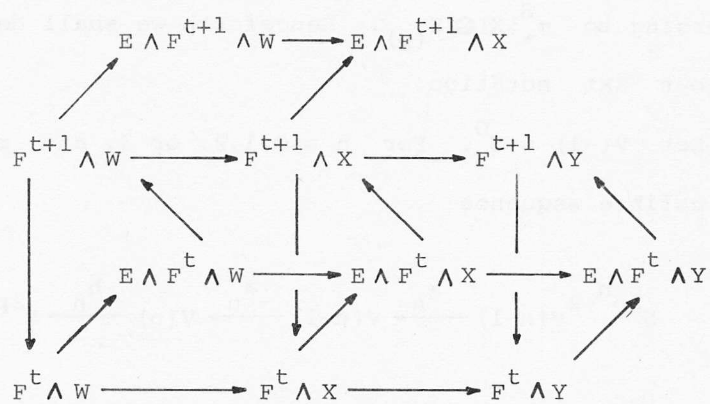
$$(j_{t+1} \wedge W)y_1 \in \pi_*^S(E \wedge F^{t+1} \wedge W)$$

represents $\delta(\bar{x})$. Because it factors through $F^{t+1} \wedge W$ it is a permanent cycle. Since $(i_s \wedge W)(F^{s+1} \wedge h) = (F^s \wedge h)(i_s \wedge Y)$ for all s, we have by (1.9) and (1.10)

$$(i_0 \cdots i_t \wedge W)y_1 = (i_0 \cdots i_{t-1} \wedge W)(F^t \wedge h)y = h \cdot (i_0 \cdots i_{t-1} \wedge Y)y = h(x).$$

That is, $\delta(\bar{x})$ represents $h(x)$.

(1.8)



2. Detecting Stable Homotopy Families.

In this section we show how the Miller-Wilson results may be combined with the geometric boundary theorem to detect stable homotopy. First we recover known results on homotopy elements of BP filtration 1 and 2; then we prove our main theorem on the gamma family.

Recall that $BP_*(\ )$ is the Brown-Peterson homology theory associated with the prime p; it has coefficient ring $BP_* \cong Z_{(p)}[v_1, v_2, \ldots]$ with $|v_n| = 2(p^n-1)$. Define $I_n = (p, v_1, \ldots, v_{n-1})$ with the convention that $v_0 = p$ and $I_0 = (0)$. All of the results of Section 1 hold for BP. In fact, there is an Adams spectral sequence

$$\mathrm{Ext}^{**}_{BP_*(BP)}(BP_*, BP_*(X)) \Longrightarrow \pi_*^S(X)$$

converging to $\pi_*^S(X) \otimes Z_{(p)}$. Henceforth we shall delete "$BP_*(BP)$" from our Ext notation.

Let $V(-1) = S^0$. For $n = 0,1,2,$ or 3, and $p > 2n$, there is a cofibre sequence

(2.1) $$S^{2p^n-2}V(n-1) \xrightarrow{\phi_n} V(n-1) \xrightarrow{a_n} V(n) \xrightarrow{h_n} S^{2p^n-1}V(n-1),$$

in which h_n is BP-proper, inducing the short exact sequence

$$0 \to BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0.$$

($n=1$: [1]; $n=2$: [10]; $n=3$: [15].)

On the E_2 level of the Adams spectral sequence, these short exact sequences induce exact triangles

$$\text{(2.2)}\qquad \begin{array}{ccc} \mathrm{Ext}^{**}(BP_*,BP_*/I_n) & \xrightarrow{v_n} & \mathrm{Ext}^{**}(BP_*,BP_*/I_n) \\ {\scriptstyle \delta_n}\nwarrow & & \swarrow{\scriptstyle \rho_n} \\ & \mathrm{Ext}^{**}(BP_*,BP_*/I_{n+1}) & \end{array}$$

where δ_n has bidegree $(1, 2-2p^n)$.

Now we need to quote two theorems.

Theorem 2.3. (Landweber [7], or see [5]) Let $n > 0$; then

$$\mathrm{Ext}^{0,*}(BP_*,BP_*/I_n) \cong \mathbb{F}_p[v_n].$$

Thus $\mathrm{Ext}^{**}(BP_*,BP_*/I_n)$ is a module over $\mathbb{F}_p[v_n]$. When $n = 0$, $\mathrm{Ext}^{0,*}(BP_*,BP_*) \cong Z_{(p)}$, concentrated in degree zero.

Theorem 2.4. (Miller-Wilson [8] passim) Let $n = 0,1,2$, or 3. If $0 \neq \rho_n(x) \in \mathrm{Ext}^{1,*}(BP_*,BP_*/I_{n+1})$, then $v_{n+1}\rho_n(x) \neq 0$.

Lemma 2.5. Let $n = 0,1,2$, or 3, and $p > 2n$. If $g_n \in \pi_s^S(V(n))$, $s > 0$, is such that

$$0 \neq BP_*(g_n) \in \mathrm{Hom}_{BP_*BP}(BP_*,BP_*/I_{n+1}) \cong \mathbb{F}_p[v_{n+1}]$$

then $0 \neq h_n g_n \in \pi_*^S(V(n-1))$. Furthermore $0 \neq h_{n-1}h_n g_n \in \pi_*^S(V(n-2))$ for $n \neq 0$.

Proof. By (2.3) the exact sequence induced by (2.2) begins, for $n = 0$,

$$0 \longrightarrow Z_{(p)} \xrightarrow{p} Z_{(p)} \xrightarrow{\rho_0} \mathbb{F}_p[v_1] \xrightarrow{\delta_0} \mathrm{Ext}^{1,*}(BP_*, BP_*) \tag{2.6}$$

and for $n > 0$;

$$0 \longrightarrow \mathbb{F}_p[v_n] \xrightarrow{v_n} \mathbb{F}_p[v_n] \xrightarrow{\rho_n} \mathbb{F}_p[v_{n+1}] \xrightarrow{\delta_n}$$

(2.7)

$$\mathrm{Ext}^{1,*}(BP_*, BP_*/I_n) \xrightarrow{v_n} \mathrm{Ext}^{1,*}(BP_*, BP_*/I_n).$$

In either case, $\delta_n(BP_*(g_n)) \neq 0$ since g_n has positive degree. By Theorem 1.7, $\delta_n(BP_*(g_n))$ is a permanent cycle converging to $h_n g_n$; see (1.6). Differentials increase homological degree by at least two, so $h_n g_n$ survives nontrivially.

Now by (2.7), $v_n \delta_n(BP_*(g_n)) = 0$. Thus by (2.4), $\delta_n(BP_*(g_n))$ cannot be in the image of ρ_{n-1} (in (2.2)). Hence, by exactness of (2.2),

$$0 \neq \delta_{n-1}\delta_n(BP_*(g_n)) \in \mathrm{Ext}^{2,*}(BP_*, BP_*/I_{n-1}) \cong E_2^{2,*}(V(n-2)).$$

By Theorem 1.7, $\delta_{n-1}\delta_n(BP_*(g_n))$ is a permanent cycle which converges to $h_{n-1}h_n g_n$; see (1.6). A glance at (2.3) shows that $\delta_{n-1}\delta_n(BP_*(g_n))$ cannot be hit by any differential so $\delta_{n-1}\delta_n(BP_*(g_n))$ survives nontrivially to E_∞ and $h_{n-1}h_n g_n \neq 0$.

Definition 2.8. $\phi_n^1 = \phi_n$ and $\phi_n^t = \phi_n \phi_n^{t-1}$, $t > 1$ (using the same symbol for a stable map and its suspension).

We now consider some examples. For $t > 0$ and $p > 2$, $\alpha_t \in \pi^S_{2(p-1)t-1}(S^0)$ is defined as the composition

$$S^{2(p-1)t} \xrightarrow{a_0} S^{2(p-1)t}V(0) \xrightarrow{\phi_1^t} V(0) \xrightarrow{h_0} S^1.$$

Notice that

$$BP_*(\phi_1^t a_0) = v_1^t \in \mathbb{F}_p[v_1] \cong \mathrm{Ext}^{0,*}(BP_*, BP_*/(p)).$$

By Lemma 2.5 we have:

Corollary 2.9 (Toda [14]). $\alpha_t \neq 0$ for all $t > 0$.

For $t > 0$ and $p > 3$, $\beta_t \in \pi^S_{2(p^2-1)t-2(p-1)-2}(S^0)$ is defined as the composition

$$S^{2(p^2-1)t} \xrightarrow{a_1 a_0} S^{2(p^2-1)t}V(1) \xrightarrow{\phi_2^t} V(1) \xrightarrow{h_1} S^{2p-1}V(0) \xrightarrow{h_0} S^{2p}.$$

Notice that

$$BP_*(\phi_2^t a_1 a_0) = v_2^t \in \mathbb{F}_p[v_2] \cong \mathrm{Ext}^{0,*}(BP_*, BP_*/(p, v_1)).$$

By Lemma 2.5 we have

Corollary 2.10 (Smith [10]). $\beta_t \neq 0$ for all $t > 0$.

<u>Remark</u> 2.11. In [8], $Ext^{1,*}(BP_*, BP_*/I_n)$ and the maps ρ_n are described completely for $n = 0,1,2,$ and 3. We have stated in (2.4) only the minimal result necessary to study the beta and gamma families. Using the more complete information availble in [8] one can use techniques similar to these to detect all of the epsilons of Oka [9], Smith [11], and Zahler [17] assuming only the spaces and self maps used in their definition.

Let $p > 5$ and $t > 0$. There are elements

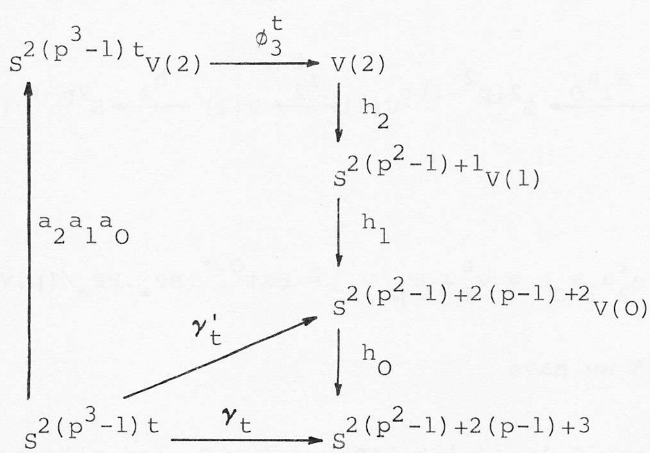
$$\gamma_t' \in \pi^S_{2(p^3-1)t-2(p^2-1)-2(p-1)-2}(V(0))$$

and

$$\gamma_t \in \pi^S_{2(p^3-1)t-2(p^2-1)-2(p-1)-3}(S^0)$$

defined by the following diagram.

(2.12)



Observe that

$$BP_*(\phi_3^t a_2 a_1 a_0) = v_3^t \in \mathbb{F}_p[v_3] \cong \mathrm{Ext}^{0,*}(BP_*, BP_*/(p,v_1,v_2)).$$

By Lemma 2.5 we have the following folk result.

<u>Corollary</u> 2.13. $\gamma_t' \neq 0$ for all $t > 0$.

In this case, the proof of Lemma 2.5 showed that

$$0 \neq \delta_1\delta_2(v_3^t) \in \mathrm{Ext}^{2,w(t)}(BP_*, BP_*/(p)),$$

where $w(t) = 2(p^3-1)t-2(p^2-1)-2(p-1)$.

<u>Lemma</u> 2.14. Suppose for some $t > 0$ that

$$\mathrm{Ext}^{2,w(t)}(BP_*, BP_*) = 0.$$

Then $\gamma_t \neq 0$.

<u>Proof</u>. With $k = w(t)$ we have

$$0 \neq \delta_1\delta_2(v_3^t) \in \mathrm{Ext}^{2,k}(BP_*, BP_*/(p))$$

and the exact sequence

$$0 = \mathrm{Ext}^{2,k}(BP_*, BP_*) \xrightarrow{\rho_0} \mathrm{Ext}^{2,k}(BP_*, BP_*/(p)) \xrightarrow{\delta_0} \mathrm{Ext}^{3,k}(BP_*, BP_*).$$

Thus $\delta_0\delta_1\delta_2(v_3^t) \neq 0$. By Theorem 1.7, this is a permanent cycle. Since $\mathrm{Ext}^{*,i}(BP_*, BP_*) = 0$ for $i \not\equiv 0$ modulo $2(p-1)$, no non-zero differential (bidegree = $(r,r-1)$) has range $\mathrm{Ext}^{3,w(t)}(BP_*, BP_*)$.

So $\delta_0\delta_1\delta_2(v_3^t)$ survives nontrivially to E_∞ and by (1.7) converges to γ_t.

We are not so lucky as to have $\mathrm{Ext}^{2,w(t)}(BP_*,BP_*) = 0$ for all t. In [6] it was observed that $\mathrm{Ext}^{2,w(1)}(BP_*,BP_*) = 0$, giving a confirmation of the theorem of Thomas and Zahler [12] that $\gamma_1 \neq 0$. From Theorem B of [8] and the discussion following it we have

<u>Theorem</u> 2.15 (Miller-Wilson). Let $p > 2$. $\mathrm{Ext}^{2,n}(BP_*,BP_*)$ is the direct sum of j nontrivial cyclic $Z_{(p)}$-modules where j is the number of times n appears in the following list.

(i) $[p^s(p+1)-i]q$ $\quad s \geq 0,\ 0 < i \leq p^s$

(ii) $[a(p+1)-1]q$

(iii) $[ap(p+1)-i]q$ $\quad 0 < i \leq p$

(iv) $[ap^s(p+1)-i]q$ $\quad s > 1,\ 0 < i \leq p^s+p^{s-1}-1$

where $1 < a,\ (a,p) = 1,\ q = 2(p-1)$.

<u>Proposition</u> 2.16. For $r = 2,3,\ldots,p-1$, $s > 0$, and $k = w(rp^s) = 2(p^3-1)rp^s-2(p^2-1)-2(p-1)$, $\mathrm{Ext}^{2,k}(BP_*,BP_*) = 0$.

<u>Proof</u>. It is easy to check that $k = w(rp^s)$ is not in the above list. For example: $k/q \equiv rp^s-1$ modulo $(p+1)$; $a(p+1)-1 \equiv -1$ modulo $(p+1)$; $rp^s \not\equiv 0$ modulo $(p+1)$ since $2 \leq r \leq p-1$; and thus k cannot be of form (ii). The other cases require equally elementary and entertaining arguments.

This gives us our main result.

Corollary 2.17. $\gamma_{rp^s} \neq 0$ for $r = 2,3,\ldots,p-1$ and $s > 0$.

Proof. (2.14) and (2.16).

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