ON NOVIKOV’S EXT1 MODULO AN INVARIANT PRIME IDEAL

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INTRODUCTION

In his work on complex cobordism [18], Novikov introduced a spectral sequence converging to the stable homotopy of a space, depending only on the complex cobordism of the space as a module over the ring of primary complex cobordism operations. This spectral sequence can be localized at a prime \( p \) and one can work, as Novikov did, with the smaller theory, \( BP^{*}(\ast) \), known as Brown–Peterson cohomology [5]. Adams [4] translated the construction into homology, and we have:

\[ E^{1}_{\ast}(X) = \text{Ext}^{\ast}_{BP\ast}(BP\ast, BP\ast(X)) \Rightarrow \pi^{\ast}_{\ast}(X)_{BP}. \]

Novikov computed the first line, \( \text{Ext}^{1}_{BP\ast}(BP\ast, BP\ast) \), of the spectral sequence for \( X \) a sphere and showed that for \( p \) odd it was canonically isomorphic with the image of \( J \). Later, Quillen’s work [19], [4] gave a good grip on the structure of the \( BP \) operations and was used by Zahler [24] for low dimensional calculations. Quillen’s results, together with Hazewinkel’s construction [6] of algebra generators for \( I_{n} = 2(p^{n} - 1) \), and the existence of the cobar construction for \( BP \) [12] (see Section 3), have now made computations with the Novikov spectral sequence really feasible.

The ideals

\[ I_{n} = (p, v_{1}, \ldots, v_{n-1}) \subseteq BP\ast, \quad 0 \leq n \leq \infty, \]

are invariant under the action of \( BP \) operations. In fact, they exhaust the list of invariant prime ideals [9]. Thus \( BP\ast/I_{n} \) is a \( BP\ast BP \) comodule, and a theorem of Landweber [9] implies that for \( 0 < n < \infty \)

\[ \text{Ext}^{1}_{BP\ast BP}(BP\ast, BP\ast/I_{n}) = F_{p}[v_{n}]. \]

The purpose of this paper is to give proofs of our calculation of

\[ \text{Ext}^{1}_{BP\ast BP}(BP\ast, BP\ast/I_{n}) = \text{Ext}^{1}_{BP\ast BP}(BP\ast, BP\ast/I_{n}). \]

These results were announced in [14].

Because of the realizability of \( BP\ast/I_{n} \) as \( BP\ast(X) \) for \( n = 1, 2, 3, 4, \) and \( p > 2(n - 1)(n = 2, [2]; \ n = 3, [21]; n = 4, [22]) \), these computations have immediate applications to stable homotopy (see [7]). In particular, the calculation of \( \text{Ext}^{1}_{BP\ast BP}(BP\ast, BP\ast/p) \) and \( \text{Ext}^{1}_{BP\ast BP}(BP\ast, BP\ast/p, v_{1}) \) are essential first steps in our work with Ravenel on computing \( \text{Ext}^{1}_{BP\ast BP}(BP\ast, BP\ast) \) and detecting the entire gamma family of stable homotopy elements [13].

In §1 we state our main results and complete the description of \( \text{Ext}^{1}_{BP\ast BP}(BP\ast, BP\ast/I_{n}) \) for an odd prime \( p \) by quoting work of Morava and Moreira. The necessary information on \( BP \) is then recalled and in Section 3 we construct the cobar complex. Next we compare \( BP\ast BP \) with the Hopf algebra dual to the reduced powers and thereby handle the case \( n = \infty \). The other cases are proved in Section 5 by constructing elements and using a downward induction on \( n \). In the last section we apply this result to classify invariant ideals containing \( I_{n} \) and generated by \( n + 2 \) elements.

Throughout this paper \( p \) will denote a prime and \( BP \) will be the Brown–Peterson spectrum associated with \( p \).

We would like to thank Dave Johnson whose joint work with the second author on \( BP \)
operations[8] led us naturally into this area of research. We also thank Raph Zahler whose early partial results on \( \text{Ext}^{1,*}(BP_\ast, BP_\ast/(p)) \) [25] stimulated our interest in the problem. We are grateful to both of the above for helping to find applications of this work to stable homotopy [7]. Special thanks go to Jack Morava whose work [15], [16] has revolutionized the field, in particular his Theorem 1.4 here showed us we had nearly complete results about \( \text{Ext}^1 \). We want to thank Doug Ravenel who pointed out to us that our proofs still worked at the prime 2 if \( n > 1 \), and with whom subsequent collaboration has made this work the first step in a much more ambitious program [13]. Both authors were partially supported by the NSF.

§1. STATEMENT OF RESULTS

Let \( I_n \) denote the invariant prime ideal \( (p, v_1, \ldots, v_{n-1}) \subset BP_\ast \). We have a short exact sequence

\[
0 \to BP_\ast/I_n \xrightarrow{u} BP_\ast/I_{n+1} \to BP_\ast/I_{n+1} \to 0
\]

which induces a 'Bockstein' exact couple (see 4.4 below)

\[
\text{Ext}^{**,}(BP_\ast, BP_\ast/I_n) \to \text{Ext}^{**,}(BP_\ast, BP_\ast/I_n)
\]

where \( \delta_n \) has bidegree \((1, 2 - 2p^n)\). Here and below we suppress \( BP_*BP \) from our notation.

\( \text{Ext}^{**,}(BP_\ast, BP_\ast/I_n) \) is a bigraded algebra (see the remarks following Lemma 3.4), so \( \text{Ext}^{1,*}(BP_\ast, BP_\ast/I_n) \) is a module over

\[
\text{Ext}^{1,*}(BP_\ast, BP_\ast/I_n) = \mathbb{F}_p[v_\ast] \quad (Z_\omega) \quad \text{if} \ n = 0.
\]

We shall describe \( \text{Ext}^{1,*}(BP_\ast, BP_\ast/I_n) \) in terms of this module structure.

Recall that \( BP_*BP = BP_\ast[I_1, I_2, \ldots] \) with \( |I_n| = 2(p^n - 1) \). In the cobar construction for \( BP_*BP \) with coefficients in \( BP_\ast/I_n \), \( n > 0 \), \( t^p \) is a cycle representing a non-zero class \( h_i \in \text{Ext}^{1,*}(BP_\ast, BP_\ast/I_n) \)

where \( q = 2(p - 1) \). Clearly \( h_i \) is taken to \( h_i \) by reduction, \( \rho_n \).

Let \( p > 2 \) and \( 0 \leq n < \infty \), or \( p = 2 \) and \( 2 \leq n < \infty \). For \( r > 0 \), write \( r = ap^s \) with \( a \) a prime to \( p \). Write \( s = kn + i + 1 \) with \( 0 \leq i < n \). If \( n = 0 \), let \( q(r) = q_0(ap^s) = s + 1 \). If \( 0 < n < \infty \), let

\[
q(r) = q_n(ap^s) = \begin{cases} p^s & \text{if } a = 1 \\ p^s + (p - 1) \sum_{i=0}^{k-1} p^{in+i} & \text{otherwise.} \end{cases}
\]

Recall that \( I_0 = (0) \) and \( v_0 = p \). In §5 we explicitly construct certain elements

\[
c_n(r) \in \text{Ext}^{1,2s(p^s+1)-1-2q(r)(p^s-1)}(BP_\ast, BP_\ast/I_n).
\]

**THEOREM 1.1.** Let \( p > 2, 0 \leq n < \infty \), or \( p = 2, 2 \leq n < \infty \),

(a) The set \( \{h_i: 0 \leq i < n\} \) generates a \( \mathbb{F}_p[v_\ast] \) submodule of \( \text{Ext}^{1,*}(BP_\ast, BP_\ast/I_n) \) of rank \( n \).

(b) The \( v_n \)-torsion submodule of \( \text{Ext}^{1,*}(BP_\ast, BP_\ast/I_n) \) is a sum of cyclic \( \mathbb{F}_p[v_n] \) \((Z_\omega)\) for \( n = 0 \) modules on generators \( c_n(r) \) for \( r = ap^s > 0 \), a prime to \( p \), satisfying

\[
(c)\quad \text{v}_n^{a(r)}c_n(r) = 0 \\
(d)\quad h_{n+1} = c_n(p^s) + v_n^{p^n-1}h_0 \\
(e)\quad \rho_n(c_n(ap^s)) = av_n^{p^n-1}h_0 \\
(f)\quad h_{n+1} = c_n(p^s) + v_n^{p^n-1}h_0 \\
(g)\quad 2av_n^{p^n-1}h_0 \\
(h)\quad av_n^{p^n-1}h_0
\]

where \( s = 1 \) \( \mod n \), \( 0 \leq i < n \).

**COROLLARY 1.2.** Let \( p > 2, 0 < n < \infty \), or \( p = 2, 1 < n < \infty \). All relations in the \( \mathbb{F}_p[v_n] \)-submodule
of $\text{Ext}^{\bullet}(BP_*, BP_*/I_n)$ generated by the $h_i$, $i \geq 0$, are consequences of

$$v_i h_{s+i} = v_i h_s, s \geq 0.$$ 

**Corollary 1.3.** Let $p > 2$, $0 \leq n < \infty$, or $p = 2$, $2 \leq n < \infty$. Suppose $I \subseteq BP_*$ is an invariant ideal containing $I_n$ and generated by $n+2$ elements and no fewer. Then $I = (p, v_1, \ldots, v_{n-1}, v_n, y)$ where $t > 0$ and $y$ involves only the generators $v_n, v_{n+1}$ and $v_{n+2}$.

**Remark.** In §6 we give a complete classification of all such ideals; this is a corollary of a description of the groups $\text{Ext}^{\bullet}(BP_*, BP_*/(I_n, v_*))$ which follows from the proof of Theorem 1.1.

In order to offer a complete description of $\text{Ext}^{\bullet}(BP_*, BP_*/I_n)$ for odd primes we summarize the work of others on the free $F_p[v_n]$ summands. In particular, we need the following important result of Morava ([15] and private communications). More recent proofs of this result have been obtained by Moreira and Ravenel. All three use different techniques.

**Theorem 1.4.** (Morava). Let $p$ be an odd prime and $0 < n < \infty$. The rank of $\text{Ext}^{\bullet}(BP_*, BP_*/I_n)$ over $F_p[v_n]$ is 1 for $n = 1$ and $n + 1$ for $n > 1$.

For $n > 1$. Theorem 1.1(a) gives us all but one generator of $\text{Ext}^{\bullet}(BP_*, BP_*/I_n)$ mod $v_n$-torsion; for $n = 1$, we have everything. To complete our description we state the main property of the remaining free summand for $n > 1$. This was conjectured by us in [14]. The actual construction of the extra generator $w_n$ is very complicated and is done explicitly by Moreira in his thesis.

**Theorem 1.5.** (Moreira[17]). For $p$ an odd prime and $1 < n < \infty$, there is an element

$$w_n \in \text{Ext}^{2n}(BP_*, BP_*/I_n)$$

where $k = 2(p^n - 1)/(p - 1)$, which generates a free $F_p[v_n]$-module summand and reduces to

$$0 \neq \rho_n(w_n) = v_n^{k+1} + \cdots + v_n^{k+n} h_{n-1}.$$ 

For example, a representing cycle for $w_2$ in the cobar construction is given by

$$v_2 t_1^{p+1} + v_1 t_1^p + v_2 t_2^2.$$ 

The main technical result we need for all of our computations concerns the right unit, $\eta_R: BP_* \to BP_* BP$.

**Lemma 5.1.** Let $n > 0$. Then

$$\eta_R(v_{n+1}) = v_{n+1} + v_n t_1^{2n} - v_n t_1 \mod I_n.$$ 

**§2. Recollections on $BP$ and $BP_*$**

We begin by reviewing the necessary facts about $BP$ (see [19] and [4]). Fix a prime $p$. There is an associative ring-spectrum $BP$ with homology algebra

$$H_*(BP_*; Z) = \mathbb{Z}(m_1, m_2, \ldots)$$

for canonical generators $m_n$, $|m_n| = 2(p^n - 1)$. Thus by the Künneth theorem,

$$H_*(BP \wedge BP_*; Z) = H_*(BP_*; Z)[t_1, t_2, \ldots]$$

where $|t_n| = 2(p^n - 1)$; and $t_n$ may be chosen inductively so that $t_0 = 1$ and

$$\eta_R(m_n) = \sum_{i+j=n} m_i t_j^{2i}.$$ 

where $\eta_R$ is induced from $BP = S^0 \wedge BP \to BP \wedge BP$. The map $BP = BP \wedge S^0 \to BP \wedge BP$ induces $\eta_L: m_n \to m_n$.

If we allow $\eta_L$ and $\eta_R$ to describe left and right $H_*(BP \wedge BP)$-module structures on $H_*(BP \wedge BP)$ then we have for any spectrum $X$ a diagram

$$H_*(BP \wedge X) \cong H_*(BP \wedge S^0 \wedge X) \to H_*(BP \wedge BP \wedge X)$$

$$\phi_X$$

$$H_*(BP \wedge BP) \otimes_{BP_* BP} H_*(BP \wedge X)$$
in which the K"unneth map $m$ is an isomorphism. In case $X = BP$, $\psi_{BP} = \Delta$ is described inductively by

$$\sum_{i,j,k=0}^{n} m_{i,j,k} (t_{i})^{p_{i}} \otimes t_{j}^{p_{j}} t_{k}^{p_{k}}.$$  \[ 2.3 \]

The switch map $BP \wedge BP \to BP \wedge BP$ induces $c$ such that

$$m_{n} = \sum_{i+j+k=n} m_{i,j,k} c (t_{i}^{p_{i}})$$  \[ 2.4 \]

and the multiplication $BP \wedge BP \to BP$ induces $\epsilon : t_{n} \mapsto 0$, $m_{n} \mapsto m_{n}$.

These maps restrict to the subalgebras

$$BP \ast \pi_{*}(BP) \subset H_{*}(BP)$$

and

$$BP \ast BP = \pi_{*}(BP \wedge BP) = BP \ast [t_{1}, t_{2}, \ldots] \subset H_{*}(BP \wedge BP).$$

Hazewinkel\[6, 26\] has shown that the elements $v_{i} \in H_{*}(BP)$ determined by

$$v_{n} = p m_{n} - \sum_{i=1}^{n-1} m_{i} v_{n-i}^{p_{i}}$$  \[ 2.5 \]

generate the subalgebra $\pi_{*}(BP) = \mathbb{Z} [v_{1}, v_{2}, \ldots]$. By convention $m_{0} = 1$, $v_{0} = p$.

3. THE COBAR CONSTRUCTION

The identities recorded in [3] which are satisfied by $\eta_{L}$, $\eta_{R}$, $\epsilon$, $\Delta$, $c$ are easily seen to be precisely the axioms for a cogroupoid object in the category of commutative graded rings. Since groupoids have bar constructions, we may expect objects such as $BP \ast BP$ to have cobar constructions, and we turn now to this device\[12\].

Fix a commutative ring $K$. Let $A$ and $\Gamma$ be commutative graded $K$-algebras equipped with $K$-algebra morphisms $\eta_{L}$, $\eta_{R}$ : $A \to \Gamma$, $\epsilon : \Gamma \to A$, $\Delta : \Gamma \to \Gamma \otimes \Gamma$, $c : \Gamma \to \Gamma$, satisfying the axioms for a cogroupoid object[3]. A $\Gamma$-comodule is a left $A$-module $M$ together with a left $A$-module map $\psi : M \to \Gamma \otimes A M$ which is associative and unitary. For example, the homotopy analogue of 2.2 makes $BP \ast BP$ a $BP \ast BP$-comodule for any spectrum $X$.

Let $\Omega^{*}(\Gamma; M) = \Gamma \otimes A \cdots \otimes A \Gamma \otimes A M$, with $n$ copies of $\Gamma$, be the $n$-th group in the cochain complex $\Omega(\Gamma; M)$ with differential

$$d(\gamma_{1} \otimes \cdots \otimes \gamma_{n} \otimes m) = 1 \otimes \gamma_{1} \otimes \cdots \otimes \gamma_{n} \otimes m$$

$$+ \frac{1}{n!} \sum_{i=1}^{n} (-1)^{i} \gamma_{1} \otimes \cdots \otimes \gamma_{i-1} \otimes \gamma_{i} \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_{n} \otimes m$$

$$+ (-1)^{n-1} \sum_{i=1}^{n} \Delta \gamma_{i} \otimes \cdots \otimes \gamma_{n} \otimes m' \otimes m''$$

where $\epsilon (i) = |\gamma_{i}| + \cdots + |\gamma_{n}| + i$ and $\Delta \gamma = \Sigma \gamma' \otimes \gamma''$, $\psi (m) = \Sigma m' \otimes m''$. Write $\Omega^{*} = \Omega(\Gamma; A)$ where the $\Gamma$-comodule structure on $A$ is given by

$$A \twoheadrightarrow \Gamma = \Gamma \otimes A A.$$  \[ 3.2 \]

Define a product

$$\Omega \Gamma \otimes \Omega(\Gamma; M) \to \Omega(\Gamma; M)$$

by juxtaposition. Then $\Omega \Gamma$ is a DG $K$-algebra and $\Omega(\Gamma; M)$ is a DG $\Omega \Gamma$-module.

Define

$$\text{Ext}^{-n}(A, M) = H^{-n} \Omega(\Gamma; M).$$

It is routine to check that this is the derived functor of $\text{Hom}_{A}(\Gamma, -)$ as described by Adams\[4\]. The DG products above make $\text{Ext}_{r}(A, A)$ a bigraded $K$-algebra and $\text{Ext}_{r}(A, M)$ an $\text{Ext}_{r}(A, A)$-module in a natural way.

In case $n = 0$, 3.1 reads $d(m) = 1 \otimes m - \Sigma m' \otimes m''$. In particular, if $M = A$,

$$d(a) = 1 \otimes a - \eta_{L}(a) \otimes 1 = \eta_{R}(a) - \eta_{L}(a)$$  \[ 3.3 \]

under the identification $\Gamma \otimes A A = \Gamma$.

We mention some immediate applications of this construction.
An ideal $I \subseteq A$ is invariant iff it is a sub-$\Gamma$-comodule under the coaction 3.2. Denote by $\Gamma/I$ the $K$-algebra $(A/I) \otimes_A \Gamma = \Gamma \otimes_A (A/I)$. Then the structure maps $\eta_I$, $\eta_\Gamma$, $\varepsilon$, $\Delta$, $c$, factor through $A/I$ and $I/I$. If $M$ is a $\Gamma$-comodule such that $IM = 0$, then $M$ is naturally a $\Gamma/I$-comodule and clearly $\Omega(\Gamma; M) = \Omega(\Gamma/I; M)$. Thus:

**Lemma 3.4.** If $I \subseteq A$ is invariant and $M$ is a $\Gamma$-comodule such that $IM = 0$, then

$$\text{Ext}_{\Gamma}(A, M) = \text{Ext}_{\Gamma/I}(A/I, M).$$

It follows that $\text{Ext}_{\Gamma}(A, A/I)$ has a natural $K$-algebra structure.

Finally, assume that $\Gamma$ is flat over $A$. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $\Gamma$-comodules. Then

$$0 \to \Omega(\Gamma; M') \to \Omega(\Gamma; M) \to \Omega(\Gamma; M'') \to 0$$

is short exact, so we obtain a long exact sequence

$$\text{Ext}_{\Gamma}(A, M') \to \text{Ext}_{\Gamma}(A, M) \to \text{Ext}_{\Gamma}(A, M'').$$

§4. More on $BP$

We return now to $BP \wedge BP$. Let $I_n$ denote the ideal $(p, v_1, \ldots, v_{n-1}) \subseteq BP$, for $0 \leq n \leq \infty$. By induction using 2.5 and 2.1 it is easy to see that $I_n$ is invariant. We are concerned with $\text{Ext}^{*,*}(BP_*^{**}, BP_*^{**}/I_n)$ where we have omitted the subscript $BP_*^{**}$.

Since $BP_* = 0 = BP_*^{**}$ unless $t = 0 \mod q = 2(p - 1)$, the cobar construction and hence also $\text{Ext}^{*,*}(BP_*^{**}, BP_*^{**}/I_n)$ exhibits the same sparseness[23] in internal degree.

In homological degree 0 we have by Novikov[18]

$$\text{Ext}^{0,*}(BP_*^{**}, BP_*^{**}) = Z_{(p^*)},$$

concentrated in degree zero and by Landweber (9); see also (8) for $0 < n < \infty$

$$\text{Ext}^{0,*}(BP_*^{**}, BP_*^{**}/I_n) = F_{p^*}[v_n].$$

Clearly

$$\text{Ext}^{0,*}(BP_*^{**}, BP_*^{**}/I_n) = F_{p^*}.$$  

By 3.5 the short exact sequence

$$0 \to BP_*^{**}/I_n \xrightarrow{v_*} BP_*^{**}/I_n \to BP_*^{**}/I_{n+1} \to 0$$

of $BP_*^{**}$-$BP$-comodules induces a long exact sequence

$$\text{Ext}^{*,*}(BP_*^{**}, BP_*^{**}/I_n) \xrightarrow{v_*} \text{Ext}^{*,*}(BP_*^{**}, BP_*^{**}/I_n) \xrightarrow{c_{BP_*^{**}}} \text{Ext}^{*,*}(BP_*^{**}, BP_*^{**}/I_{n+1}).$$

These long exact sequences will be the basis of an inductive calculation in §5. For $n = \infty$, we need the following result.

**Lemma 4.5.** In $BP_*^{**}$, $BP_*^{**} = \bigoplus_{i+j=n} t_i \otimes t_j^p$ mod $I_n$.

**Proof.** From 2.5 it follows that $pH_*^{**}(BP') \cap \pi_*^{**}(BP') = I_n$. Therefore $\pi_*(BP)/I_n \cong H_*(BP; Z/pZ)$, and similarly for $BP \wedge BP$ and $BP \wedge BP \wedge BP$. Hence we calculate in $H_*(BP \wedge BP; Z/pZ)$. 

\[\Delta t_n = \sum_{i+j=n} t_i \otimes t_j^p \mod I_n.\]
Clearly $\Delta t_i = t_i \otimes 1 + 1 \otimes t_i$. By induction using 2.3,

$$\Delta t_n = \sum_{i+j=k-n} m_it_i^{p^r} \otimes t_j^{p^r} - \sum_{i+j=n} m_i(\Delta t_i)^{p^r}$$

$$= \sum_{i+j=k-n} m_i \sum_{i+k-h} t_i^{p^r} \otimes t_k^{p^r} - \sum_{i+k-h} m_i \left( \sum_{i+j=h} t_j \otimes t_k^{p^r} \right)^{p^r}$$

$$= \sum_{i+j=k-n} t_i \otimes t_k^{p^r}.$$

From Lemma 3.4, then:

**Corollary 4.6.** Let $P_\ast = F_\ast \{t_1, t_2, \ldots\}$ with diagonal $\Delta t_n = \sum_{i+j=n} t_i \otimes t_j^{p^r}$. Then

$$\text{Ext}^{\ast\ast}(BP_\ast, BP_\ast/I_\ast) = \text{Ext}^{\ast\ast}(F_\ast, F_\ast).$$

Note that $P_\ast$ is just the Hopf algebra dual to the Steenrod reduced powers. Indeed, in [24] Zahler proved that the Thom map $BP_\ast BP \rightarrow HF_\ast \rightarrow A_\ast$ carries $t_i$ to $c(\xi_i^p)$ for $p > 2$ and to $c(\xi_i^2)$ for $p = 2$, where $c$ is the antiautomorphism of the dual Steenrod algebra $A_\ast$.

The work of Adem as interpreted by Adams [1] and Liulevicius [11] now implies:

**Corollary 4.7.** $\text{Ext}^{\ast\ast}(BP_\ast, BP_\ast/I_\ast)$ is the $F_\ast$-module generated by the classes $h_i$ of $t_i^{p^r} \in \Omega(BP_\ast BP; BP_\ast/I_\ast)$, $i \geq 0$.

§ 5. THE TORSION SUBMODULE

Our purpose in this section is to exhibit certain cycles in $\Omega^i(BP_\ast BP; BP_\ast/I_\ast)$ and to prove that their classes generate the torsion $F_\ast[v_i]$-submodule of $\text{Ext}^{\ast\ast}(BP_\ast, BP_\ast/I_\ast)$. We begin with our basic technical lemma.

**Lemma 5.1.** For $0 < k < \infty$,

$$\eta_\ast(v_{n+1}) = v_{n+1} + \varepsilon v_{n+1}^{p^r} - v_n v_{t_1} \text{ mod } I_\ast.$$ 

**Proof.** Note first by induction using 2.5 that $p^i m_i \in (v_1, v_2, \ldots, v_{n-1})^\pi(BP)$ for $0 < i < n$.

Thus if $x \in (m_1, \ldots, m_{n-1})^\pi(BP)$, then $p^i x \in (v_1, \ldots, v_{n-1})^\pi(BP)$. Since $\pi(BP)/(v_1, \ldots, v_{n-1})$ is free of $p$-torsion, this implies that $x \in (v_1, \ldots, v_{n-1})^\pi(BP)$.

So we compute in $H_\ast(BP \otimes BP)$ modulo $(m_1, \ldots, m_{n-1})^\pi(BP)$. From 2.5 it is immediate that $pm_n = v_n, pm_n^{p^r} = v_n^{p^r}$, and $v_i \equiv 0, 0 < i < n$. Now using 2.1, 2.5, and the fact that $\eta_\ast$ is a ring homomorphism,

$$\eta_\ast(v_{n+1}) = p\eta_\ast(m_{n+1}) - \sum_{i=1}^n \eta_\ast(m_i)\eta_\ast(v_{n+1}^p)$$

$$= p \sum_{j=0}^{n+1} n_j^{p^r} - \sum_{i=1}^n m_i t_i^{p^r} \eta_\ast(v_{n+1}^p)$$

$$= \eta_\ast(v_{n+1}) + \eta_\ast(m_n v_n v_{t_1}^{p^r}) + \eta_\ast(m_n - m_n v_n v_{t_1}^{p^r}) - \sum_{i=1}^n l_i \eta_\ast(v_{n+1}^p).$$

Now $pm_1 = v_1$ and $\eta_\ast(v_1) = v_1 + pt_1$, so for $n > 1$,

$$m_n \eta_\ast(v_{n+1}) = v_n p^{n-1} t_1^{p^r} + m_n v_1^{p^r}.$$ 

We leave the easier case $n = 1$ to the reader. Thus for $n > 1$,

$$\eta_\ast(v_{n+1}) = pt_{n+1} + v_n t_1^{p^r} + v_{n+1} - p^{n-1} v_{n+1} t_1^{p^r} - \sum_{i=1}^n l_i \eta_\ast(v_{n+1}^p).$$

This holds mod $(v_1, \ldots, v_{n-1}) \subset I_\ast$ by the above remarks; so continuing mod $I_\ast$,

$$\eta_\ast(v_{n+1}) = v_n^{p^r} + v_{n+1} - l_1 v_n^{p^r}.$$ 

Here we have used the fact that $\eta_\ast(v_k) = v_k \text{ mod } I_\ast$.

**Remark.** D. C. Ravenel[20] has proved a beautiful generalization of this lemma, and we
expect his formulas to be very useful. For example, they are essential in Moreira's proof of Theorem 1.5.

Our search for cycles representing torsion generators of the $F_n[v_n]$-module $\text{Ext}^{i,*}(BP_*, BP_*/I_n)$ is governed by the following immediate consequence of 4.2 and 4.4.

**Lemma 5.2.** Let $0 \leq n < \infty$. The elements $\delta_i(v_n^{i+1})$, $i > 0$, are nonzero and generate

$$\ker v_n \subset \text{Ext}^{i,*}(BP_*, BP_*/I_n)$$

additively.

Since every connected module over $F_n[v_n]$ is a direct sum of cyclic modules, we must find how far $\delta_i(v_n^{i+1})$ can be divided by $v_n$. Notice that since $\Omega^i(BP_*/BP_*/I_n)$ is a DG $F_n[v_n]$-module (if $n = 0$, a DG $\mathbb{Z}(p)$-module) which is free of $v_n$-torsion, $c$ is a cycle iff $\bar{c} = v_n^s c$ is a cycle for some $k$.

If $p = 2$, the cases $n = 0, 1$ behave peculiarly and will be treated elsewhere in joint work with D. C. Ravenel; see also Remark 5.10 below. The case $p = 2$, $n = 0$ was solved by Novikov [18]. So assume that either $p > 2$ and $0 \leq n < \infty$ or that $p = 2$ and $2 \leq n < \infty$.

Let $s = kn + i + 1$ with $0 \leq i < n$. For a prime to $p$ and $1 \leq j \leq k$ let

$$a_i = a_i(n, ap^*) = (ap - 1)p^m + 1$$

$$b_j = b_j(n, ap^*) = \sum_{i=0}^{k-1} p^{(k-1)n + i + 1} - \sum_{j=1}^{n-1} p^{(k-1)n - j}.$$  

For $r = ap^* > 0$, a prime to $p$, define

$$\tilde{c}_n(ap^*) = v_n^{ap^*}$$ if $n = 0$ or $s = 0$ or $a = 1$;

$$\tilde{c}_n(ap^*) = v_2^{ap^* - 1} - av_1^{ap^* - 1}v_2^{(a-1)p^r - 1}v_5^{p^r - 1} - av_1^{b_r}v_2^{n - 2}$$

$$= \tilde{u}_2^{n + ap^* - 1 - p^r - 1}v_2^{(a-1)p^r - 1}v_5^{p^r - 1} - 2a \sum_{j=1}^{n-2} v_1^{b_r}v_2^{n - 2}$$

if $n = 1$, $s > 1$, and $a \neq 1$;

$$\tilde{c}_n(ap^*) = v_n^{ap^* - 1} - av_1^{ap^* - 1}v_2^{(a-1)p^r - 1}v_5^{p^r - 1} - a \sum_{j=1}^{n-2} v_1^{b_r}v_2^{n - 1}$$ otherwise.

If $n = 0$ let $q(ap^*) = q_0(ap^*) = s + 1$. If $0 < n < \infty$, let

$$q(ap^*) = q_n(ap^*) = \begin{cases} p^r & \text{if } a = 1, \\ p^r + (p - 1) \sum_{i=0}^{k-1} p^{m + i} & \text{otherwise.} \end{cases}$$

**Proposition 5.3.** Let $p > 2$, $0 \leq n < \infty$, or $p = 2$, $2 \leq n < \infty$, and let $r > 0$.

(a) There exists a unique cycle

$$c_n(r) \in \Omega^i(BP_*/BP_*/I_n)$$

such that

$$v_n^{\hat{n}(r)}c_n(r) = d\tilde{c}_n(r).$$

(b) Modulo $I_{n+1}$ we have, for a prime to $p$,

$$c_n(ap^*) = av_1^{ap^* - 1}t_1$$

if $n = 0$

$$= t_1^{p^r}t_1$$

if $a = 1$

$$= av_2^{ap^* - 1}t_1^{p^r}$$

if $s = 0$

$$= 2av_2^{ap^* - 1}t_1^{p^r}$$

if $n = 1, s > 1$, and $a \neq 1$

$$= av_n^{ap^* - 1}t_1^{p^r}$$

otherwise

where $s - 1 = i \mod n$, $0 \leq i < n$.

**Proof.** We work mod $I_n$. Recall that for $x, y \in BP_*/I_n$, $d(x) = \eta_R(x) - \eta_L(x) = \eta_R(x) - x \mod I_n$ by 3.3, and that $\eta_R(xy) = \eta_R(x)\eta_R(y)$. In case $n = 0$ we have $\eta_R(v_1) = v_1 + pt_1$, and the result is straightforward. In case $s = 0$ or $a = 1$, both assertions are clear from the congruence.
\[ dv_n^{p^s} = av_n^{p^s} v_n^{(a-1)p^s} t_1^{p^{s+n}} \mod v_n^{2p^s}. \]

which is a consequence of Lemma 5.1.

The other cases result similarly from the following consequences of 5.1:

\[ d(v_n^{(a-1)p^s+1-p}) = v_n^{a(p_{n+1})^s} t_1^{p^{s+n}} - v_n^{(a-1)p^s+1} t_1^{p^{s+n}} \mod v_n^{p^{s+1}}. \]

\[ d(v_n^{(a-1)p^s+1-p}) = v_n^{a(p_{n+1})^s} t_1^{p^{s+n}} - v_n^{(a-1)p^s+1} t_1^{p^{s+n}} \mod v_n^{p^{s+1}}. \]

\[ d(v_n^{(a-1)p^s+1-p}) = v_n^{a(p_{n+1})^s} t_1^{p^{s+n}} - v_n^{(a-1)p^s+1} t_1^{p^{s+n}} \mod v_n^{p^{s+1}}. \]

**Remark.** If 0 < s \leq n, a > 1, than \( dv_n^{p^s} \) and \( d\tilde{c}_n(a^{p^s}) \) are divisible by the same power of \( v_n \), but \( v_n^{p^s} \) \( d\tilde{c}_n(a^{p^s}) \) has a more convenient value modulo \( I_n \).

**Remark 5.9.** Thus

\[ \tilde{c}_n(r) \in \text{Ext}_{\nu}^s(BP_*, BP_*/(I_n, v_n^{q(r)})) \subset BP_*(I_n, v_n^{q(r)}), \]

and \( c_n(r) \in \text{Ext}_*^t(BP_*, BP_*/I_n) \) is the image of \( \tilde{c}_n(r) \) under the boundary homomorphism induced by the short exact sequence

\[ 0 \to BP_*/I_n \xrightarrow{v_n^{q(r)}} BP_*/I_n \to BP_*(I_n, v_n^{q(r)}) \to 0. \]

By customary abuse of notation let \( c_n(r) \) denote the class in \( \text{Ext}_*^t(BP_*, BP_*/I_n) \) represented by the cycle \( c_n(r) \). Recall the class \( h_n \) of \( t_1^{p^s} \) in \( \text{Ext}_*^{2(p^s-1)}(BP_*, BP_*/I_n) \), 0 < n \leq \infty. We can now prove the main theorem stated in §1.

**Proof of Theorem 1.1**

First note that properties (i) and (iii) follow from Proposition 5.3 and Lemma 5.2. Property (ii) holds on the cochain level:

\[ c_n(p^s) = t_1^{p^{s+n}} - v_n^{p^{s+1-p}} t_1^{p^s} \]

by Lemma 5.1.

Suppose now \( n > 0 \). Let \( M(n) \) be the \( F_\nu[v_\nu] \)-module with generators \( h_i', 0 \leq i < n \) and \( c_n'(r) \), 0 < r, subject only to \( v_n^{a^{p^s}} c_n'(r) = 0 \). By (i) there is a map

\[ f_n : M(n) \to \text{Ext}_*^t(BP_*, BP_*/I_n) \]

carrying \( h_i' \) to \( h_i \) and \( c_n'(r) \) to \( c_n(r) \). We must show that \( f_n \) is injective and has torsion-free cokernel.

To show \( f_n \) is injective, we must show \( \text{Ext}_*^t(BP_*, BP_*/I_n) \) is (2(p^s - 1)) connected.

By customary abuse of notation let \( c_n(r) \) denote the class in \( \text{Ext}_*^t(BP_*, BP_*/I_n) \) represented by the cycle \( c_n(r) \). Recall the class \( h_n \) of \( t_1^{p^s} \) in \( \text{Ext}_*^{2(p^s-1)}(BP_*, BP_*/I_n) \), 0 < n \leq \infty. We can now prove the main theorem stated in §1.

**Proof of Theorem 1.1**

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\[ f_n : M(n) \to \text{Ext}_*^t(BP_*, BP_*/I_n) \]

carrying \( h_i' \) to \( h_i \) and \( c_n'(r) \) to \( c_n(r) \). We must show that \( f_n \) is injective and has torsion-free cokernel.

The (2(p^s - 1)) connected map \( BP_*/I_n \to BP_*/I_n = F_\nu \) induces for each \( s \geq 0 \) a map \( \Omega^t(BP_*/BP_*/I_n) \to \Omega^t(BP_*/BP_*/I_n) \) with the same connectivity. Thus

\[ \text{Ext}_*^t(BP_*, BP_*/I_n) \to \text{Ext}_*^t(BP_*, BP_*/I_n) \]

is (2(p^s - 1)) connected.

Fix \( L > 0 \) and let \( n \) be such that \( 2(p^s - 1) > L \). Then in internal degrees \( t < L \), Theorem 1.1 is equivalent to Corollary 4.7. We now proceed by downward induction on \( n \).

Recall that for an \( F_\nu[v_\nu] \)-module \( M \) we have an exact sequence

\[ 0 \to \text{Tor}(F_\nu, M) \to M \xrightarrow{v_n} M \to F_\nu \otimes F_\nu[I_n] \to 0. \]

Observe that \( f_n : M(n) \to \text{Ext}_*^t(BP_*, BP_*/I_n) \) is injective with torsion-free cokernel in degrees \( t < L \). if simultaneously \( \text{Tor}(F_\nu, M) \) is surjective and \( F_\nu \otimes F_\nu[I_n] f_n \) is injective in degrees \( t < L \).
So assume that $f_{n+1}$ injects in degrees $t < L$. Consider

$$M(n) \xrightarrow{f_n} \text{Ext}^t(BP_*, BP_*/I_n)$$

$$\Gamma_{\rho_n} \otimes_{\mathbb{F}_p[I_n]} M(n) \xrightarrow{f_n} \mathbb{F}_p[I_n] \otimes_{\mathbb{F}_p[I_n]} \text{Ext}^t(BP_*, BP_*/I_n)$$

$$\Gamma_{\rho_n} \otimes_{\mathbb{F}_p[I_n]} M(n) \xrightarrow{f_n} \text{Ext}^t(BP_*, BP_*/I_{n+1})$$

where $\rho_n$ factors $\rho_n$ and $\rho_n'$ is the obvious map given by (ii) and (iii). Then $\rho_n'$ is injective, so by the inductive assumption $\rho_n f_n = f_{n+1} \rho_n'$ is injective. Thus $f_n = \mathbb{F}_p \otimes_{\mathbb{F}_p[I_n]} f_n$ injects. Since $\text{Ext}^t(BP_*, BP_*/I_n)$ is exact and the left group is $\mathbb{F}_p[v_{n+1}]$, (i) implies that $\text{Tor}(\mathbb{F}_p, f_n)$ surjects. So the induction is complete. Since $L$ was arbitrary this finishes the proof for $n > 0$.

For $n = 0$ it suffices to show that $\rho_n (r) \neq 0$ for $r > 0$. This follows from (iii) and from (a) for $n = 1$. It is elementary that there is no torsion-free part.

Remark. As one can see by the simplicity of the proof, the work is mostly done in the finding of the numbers $q_n(r)$ and the cycles $c_n(r)$. At the request of the referee we comment on how we arrived at these formulas. Our insight came from computing $\text{Ext}^t(BP_*, BP_*/(p))$ for $p = 3$ up through dimension 7776.

Remark 5.10. In case $p = 2$, $n = 1$, let

$$q(2^a) = \begin{cases} 2^s & \text{if } s = 0 \text{ or } s = 1 \text{ or } a = 1, \\ 2^s + 2^{s-1} & \text{otherwise} \end{cases}$$

for $a$ odd.

Then Proposition 5.3(a) clearly still holds, and in joint work with D. C. Ravenel we will show that Theorem 1.1(a) and (b) i, ii hold too.

§6. INVARIANT IDEALS

Recall that $I \subset BP_*$ is invariant iff it is a sub $BP_*$-comodule. Our goal in this section is to classify all invariant ideals of the form $(I, x, y)$, $n > 0$, for $p$ odd and $n \geq 2$ for $p = 2$.

Computations of Ext groups are useful in this direction because of the following observation.

Lemma 6.1. If $I$ is invariant and $x \in BP_*$, then $(I, x)$ is invariant iff

$$\bar{x} \in \text{Ext}^a(BP_*, BP_*/I) \subset BP_*/I$$

where $\bar{x}$ is $x$ mod $I$.

Clearly $(I, x) = (I, x)$ iff $x = uy$ mod $I$ for some unit $u$, so ideals of the form $(I, x)$ are classified by $\text{Ext}^a(BP_*, BP_*/I)$ modulo units.

An easy induction proves:

Lemma 6.2. Let $I \subset BP_*$ be any proper ideal containing $I_n$ and minimally generated by \{x_0, \ldots, x_k\} where $|x_0| \leq \cdots \leq |x_k|$. Then $k \geq n - 1$ and $I = (p, v_1, \ldots, v_{n-1}, x_0, \ldots, x_k)$.

Our primary interest is in invariant ideals containing $I_n$ generated by $n + 2$ elements and no fewer. By 6.2 they are of the form $(p, v_1, \ldots, v_{n-1}, x, y)$ with $|x| \leq |y|$. For dimensional reasons $(p, v_1, \ldots, v_{n-1}, x)$ is invariant; and by the above discussion,

$$\bar{x} \in \text{Ext}^a(BP_*, BP_*/I_n) \Rightarrow \mathbb{F}_p[v_n],$$

so we may take $x$ to be $v_{n+t}$, for some $t > 0$. Thus the problem of classifying all invariant ideals
$(I_n, x, y)$ is reduced to classifying invariant ideals of the form $(I_n, v_0, y)$. Again, by the above discussion, the following computation will allow us to complete our classification.

**Proposition 6.3.** Let $p > 2$ and $n \geq 0$ or $p = 2$ and $n \geq 2$. Then $\text{Ext}^*(\mathcal{B}_p, \mathcal{B}_p/(I_n, v_0))$ is the sub-$F_p[v_n]/(v_n^0(p^0))$ if $n = 0$) module of $\mathcal{B}_p/(I_n, v_0)$ generated by $1$ and $v_n^0 \varepsilon_n(r)$ where $m = \max(0, t - q_n(r))$, $r > 0$.

**Remark.** As will be clear from our proof, these elements in fact generate summands. $1$ generates a free $F_p[v_n]/(v_n^0(p^0))$ if $n = 0$) submodule, as does $\varepsilon_n(r)$ if $q_n(r) \geq t$. If $q_n(r) < t$ then $v_n^0 \varepsilon_n(r)$ generates a submodule of the form $F_p[v_n]/(v_n^0, v_n^0(p^0))$ if $n = 0$).

**Proof.** From the long exact sequence of

$$0 \rightarrow \mathcal{B}_p/I_n \xrightarrow{v_0^t} \mathcal{B}_p/I_n \rightarrow \mathcal{B}_p/(I_n, v_0) \rightarrow 0$$

we obtain a ‘universal coefficient’ short exact sequence

$$0 \rightarrow \text{coker}(v_0^t|\text{Ext}^*(\mathcal{B}_p, \mathbb{Z}_p/I_n)) \rightarrow \text{Ext}^*(\mathcal{B}_p, \mathcal{B}_p/(I_n, v_0)) \rightarrow \text{ker}(v_0^t|\text{Ext}^*(\mathcal{B}_p, \mathcal{B}_p/I_n)) \rightarrow 0.$$

The first term provides the submodule generated by 1. By Theorem 1.1, the last term is generated as an $F_p[v_n]/(v_n^0(p^0))$ if $n = 0$) module by

$$\{v_n^0 \varepsilon_n(r): k = \max(0, q_n(r) - t), r > 0\}.$$ 

Now the result follows from the construction of $\varepsilon_n(r)$; see Remark 5.9.

These Ext groups now figure in the following classification theorem.

**Theorem 6.4.** Let $n \geq 0$.

(a) All invariant ideals containing $I_n$ and generated by $n + 2$ elements and no fewer are of the form

$I = (p, v_1, \ldots, v_{n-1}, v_n, y)$

with $t > 0$, $|v_n| \leq |y|$, and

$$0 \neq \bar{y} \in \text{Ext}^0((\mathcal{B}_p, \mathcal{B}_p/(I_n, v_n)) \subset \mathcal{B}_p/(I_n, v_n),$$

where $\bar{y}$ is $y$ mod $(I_n, v_n)$.

(b) $I = I'$ iff $t = t'$ and $\bar{y} = u\bar{y}'$, $u$ a unit.

(c) $I$ is a regular ideal iff $y \neq 0$ mod $I_{n+1}$.

**Proof.** The first two parts are proved in discussion throughout the section. Their worth comes from the computation 6.3. Part (c) is straightforward.

The proof of Corollary 1.3 in the introduction follows from 6.3, 6.4, and the construction of the elements $\varepsilon_n(r)$.

See [10] for a more general discussion of invariant regular ideals.

**REFERENCES**


*Princeton University and Harvard University*
*Institute for Advanced Study and Princeton University*