HOPF RINGS IN THE BAR SPECTRAL SEQUENCE

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1. Introduction

OUR results are concerned with a very general algebraic pairing associated with the bar spectral sequence for Ω-spectra. To illustrate our main result, let $\mathcal{G}_* = \{\mathcal{G}_k\}_{k\in\mathbb{Z}}$ be a connective multiplicative Ω-spectrum. Recall that $\Omega\mathcal{G}_{k+1} \simeq \mathcal{G}_k$ and we are given maps $\circ$: $\mathcal{G}_k \wedge \mathcal{G}_n \to \mathcal{G}_{k+n}$. We are interested in studying $E_*\mathcal{G}_* = \{E_*\mathcal{G}_k\}_{k\in\mathbb{Z}}$, where $E_*(-)$ is a multiplicative homology theory. If $E_*(-)$ has a Künneth isomorphism for the spaces $\mathcal{G}_*$, then $E_*\mathcal{G}_*$ is a "Hopf ring", i.e., a ring object in the category of E_* coalgebras (see [**4**]). Our main analysis is of how the pairing

$$\mathcal{G}_{k+1} \wedge \mathcal{G}_n \to \mathcal{G}_{k+n+1} \tag{1.1}$$

behaves with respect to the bar construction. Let $\mathcal{G}'_i$ be the zero component of $\mathcal{G}_i$. Each $\mathcal{G}'_{i+1}$ is the bar construction of $\mathcal{G}_i = \Omega\mathcal{G}'_{i+1}$, the Moore loops on $\mathcal{G}'_{i+1}$, i.e., $B\mathcal{G}_i \simeq \mathcal{G}'_{i+1}$ is B on the Moore loops $\Omega\mathcal{G}'_{i+1}$. Let $F^sB\mathcal{G}_i$ be the bar filtration for $\mathcal{G}'_{i+1}$, then the map 1.1 preserves filtration in the sense that

$$F^sB\mathcal{G}_k \wedge \mathcal{G}_n \to F^sB\mathcal{G}_{k+n}. \tag{1.2}$$

Since the bar spectral sequence

$$E^r_{*,*}(E_*\mathcal{G}_k) \Rightarrow E_*\mathcal{G}'_{k+1}$$

is a spectral sequence arising from filtered spaces ([**6**], [**7**]), 1.2 implies that there is a pairing

$$\circ\colon\ E^r_{s,*}(E_*\mathcal{G}_k) \otimes_{E_*} E_*\mathcal{G}_n \to E^r_{s,*}(E_*\mathcal{G}_{k+n}) \tag{1.3}$$

with $d^r(x\circ y) = d^r(x)\circ y$. This pairing is compatible with the map

$$\circ\colon\ E_*\mathcal{G}_{k+1} \otimes_{E_*} E_*\mathcal{G}_n \to E_*\mathcal{G}_{k+n+1}.$$

Recall that

$$F^sB\mathcal{G}_k/F^{s-1}B\mathcal{G}_k \simeq \Sigma^s \wedge \underbrace{\mathcal{G}_k \wedge \cdots \wedge \mathcal{G}_k}_{s\text{-copies}}. \tag{1.4}$$

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A more detailed analysis of 1.1 reveals that the map

$$
\begin{array}{ccc}
(F^sB\mathcal{G}_k/F^{s-1}B\mathcal{G}_k)\wedge\mathcal{G}_n & \longrightarrow & F^sB\mathcal{G}_{k+n}/F^{s-1}B\mathcal{G}_{k+n} \\
\downarrow\simeq & & \downarrow\simeq \\
\Sigma^s\wedge\underbrace{\mathcal{G}_k\wedge\cdots\wedge\mathcal{G}_k}_{s\text{-copies}}\wedge\mathcal{G}_n & \longrightarrow & \Sigma^s\wedge\underbrace{\mathcal{G}_{k+n}\wedge\cdots\wedge\mathcal{G}_{k+n}}_{s\text{-copies}}
\end{array}
\tag{1.5}
$$

is induced in the obvious way by the pairing

$$\mathcal{G}_k\wedge\mathcal{G}_n\to\mathcal{G}_{k+n}.$$

If the $E_*(-)$ Künneth isomorphism holds for spaces $\mathcal{G}_*$, then $E^1_{*,*}(E_*\mathcal{G}_k)$ is the bar construction on $\tilde{E}_*\mathcal{G}_k$. Furthermore $E^2_{s,t}(E_*\mathcal{G}_k)\cong \mathrm{Tor}^{E_*\mathcal{G}_k}_{s,t}(E_*,E_*)$ and the spectral sequence is one of Hopf algebras. Thus the map 1.3 is quite easy to compute on E^1 and E^2 using 1.5. In this case the pairing is compatible with the Hopf ring structure.

This type of pairing was first observed in **[5]** for the case $\mathcal{G}_*=\mathcal{K}(\mathbb{Z}/(p^k),*)$, the mod p^k Eilenberg-MacLane spectra. The use of this pairing is necessary for the computation of the Morava K-theories of Eilenberg-MacLane spaces. The Künneth isomorphism always holds for the Morava K-theories, $K(n)_*(-)$, ([**2**], [**5**]), and for standard mod p singular homology. In fact, the mod p homology of the Eilenberg-MacLane spaces can be computed using this pairing without any need to use chains or cohomology operations. The Künneth isomorphism can be obtained in other situations as well; for example, if $\mathcal{G}_*$ is the Ω-spectrum associated with complex cobordism, then there is a Künneth isomorphism for these spaces for any complex orientable $E_*(-)$.

We continue with the same notation and definitions as above for the statement of our theorems.

THEOREM 1.6 *Let $\mathcal{A}\wedge\mathcal{C}\to\mathcal{F}$ be a pairing of connective spectra.*

(i) There exists the following homotopy commutative diagram:

$$
\begin{array}{ccc}
F^sB\mathcal{A}_k\wedge\mathcal{C}_n & \longrightarrow & F^sB\mathcal{F}_{k+n} \\
\downarrow & & \downarrow \\
B\mathcal{A}_k\wedge\mathcal{C}_n & \longrightarrow & B\mathcal{F}_{k+n} \\
\downarrow\simeq & & \downarrow\simeq \\
\circ\colon\ \mathcal{A}'_{k+1}\wedge\mathcal{C}_n & \longrightarrow & \mathcal{F}'_{k+n+1}
\end{array}
$$

(ii) The map

$$
\begin{array}{ccc}
(F^sB\mathcal{A}_k/F^{s-1}B\mathcal{A}_k)\wedge\mathcal{C}_n & \longrightarrow & F^sB\mathcal{F}_{k+n}/F^{s-1}B\mathcal{F}_{k+n} \\
\downarrow\simeq & & \downarrow\simeq \\
\Sigma^s\wedge\underbrace{\mathcal{A}_k\wedge\cdots\wedge\mathcal{A}_k}_{s\text{-copies}}\wedge\mathcal{C}_n & \longrightarrow & \Sigma^s\wedge\underbrace{\mathcal{F}_{k+n}\wedge\cdots\wedge\mathcal{F}_{k+n}}_{s\text{-copies}}
\end{array}
$$

is induced in the obvious way by

$$\circ\colon \mathscr{A}_k \wedge \mathscr{C}_n \to \mathscr{F}_{k+n}.$$

This geometric result implies our main theorem on the bar spectral sequence.

THEOREM 1.7. *Let $\mathscr{A} \wedge \mathscr{C} \to \mathscr{F}$ be a pairing of connective spectra.*
(i) There is a pairing

$$\circ\colon E^r_{s,*}(E_*\mathscr{A}_k) \otimes_{E_*} E_*\mathscr{C}_n \to E^r_{s,*}(E_*\mathscr{F}_{k+n}),$$

with $d^r(x \circ y) = d^r(x) \circ y$ and $E^r_{*,*}(E_*\mathscr{G}_k) \Rightarrow E_*\mathscr{G}'_{k+1}$ the bar spectral sequence. This is compatible with

$$\circ\colon E_*\mathscr{A}_{k+1} \otimes_{E_*} E_*\mathscr{C}_n \to E_*\mathscr{F}_{n+k+1}.$$

(ii) If $E^1_{s,*+s}(E_*\mathscr{G}_t) \simeq \tilde{E}_*(\Sigma^s \wedge \underbrace{\mathscr{G}_t \wedge \cdots \wedge \mathscr{G}_t}_{s\text{-copies}})$ is isomorphic to

$$\tilde{E}_*(\Sigma^s) \otimes_{E_*} \otimes^s_{E_*} \tilde{E}_*(\mathscr{G}_t) \quad \text{for} \quad \mathscr{G}_t = \mathscr{A}_k, \mathscr{C}_n, \quad \text{and} \quad \mathscr{F}_{k+n},$$

and $\tilde{\psi}^s$ is the iterated reduced coproduct with $\tilde{\psi}^s(x) = \sum x^{(1)} \otimes \cdots \otimes x^{(s)}$, $x \in E_*\mathscr{C}_n$, then the map of (i),

$$\circ\colon E^1_{s,*}(E_*\mathscr{A}_k) \otimes_{E_*} E_*\mathscr{C}_n \to E^1_{s,*}(E_*\mathscr{F}_{k+n}),$$

is given by

$$(y_1 \otimes \cdots \otimes y_s) \circ x = \sum \pm (y_1 \circ x^{(1)}) \otimes \cdots \otimes (y_s \circ x^{(s)})$$

where the $\circ$ on the right is for

$$\circ\colon E_*\mathscr{A}_k \otimes_{E_*} E_*\mathscr{C}_n \to E_*\mathscr{F}_{k+n},$$

and the signs are computed from the usual conventions.

Although the result seems to be new, the proof of Theorem 1.6 is really fairly simple. In fact, when shown the result, most experts quickly produce their own proof. Our original proof used Segal's machine built pairings. To do so it was necessary to show all pairings arise in this fashion. The referee has produced a far more transparent proof, and it is the referee's proof that we present in this paper. We thank the referee on behalf of the readers as well as ourselves for allowing us to use his proof. The referee also observes that the spectrum $\mathscr{C}$ in 1.6 and 1.7 can be replaced by a space. This is clear in the proof.

Despite the simplicity of the proof, the power of the pairing, as demonstrated in [**5**], is enormous. This power motivated our presenting this more general result in hopes that by making it available it will lead to even richer applications. In fact, a number of applications are already in progress.

2. Proofs

The proof of Theorem 1.6 consists of a series of simple observations. Let $\mathscr{A} \wedge \mathscr{C} \to \mathscr{F}$ be a pairing of spectra. There is then a map

$$\mathscr{A}_{k+1} \wedge \mathscr{C}_n \to \mathscr{F}_{n+k+1} \tag{2.1}$$

If Ω is the Moore loop functor, there is an induced map

$$(\Omega \mathscr{A}'_{k+1}) \wedge \mathscr{C}_n \to \Omega(\mathscr{A}'_{k+1} \wedge \mathscr{C}_n) \to \Omega \mathscr{F}'_{n+k+1} \tag{2.2}$$

The first map in the sequence sends a loop in $\mathscr{A}'_{k+1}$ and a point c in $\mathscr{C}_n$ to the loop in the smash product that is the image of the loop in $\mathscr{A}'_{k+1} \times \mathscr{C}_n$ that lies over the original loop and is contained in $\mathscr{A}'_{k+1} \otimes c$. For each c in $\mathscr{C}_n$, the map

$$\Omega \mathscr{A}'_{k+1} \to \Omega \mathscr{F}'_{n+k+1}$$

induced by the map (2.2) at level c is a map of topoligical monoids. Thus the map (2.2) induces a map of simplicial topoligical spaces from the smash product of $\mathscr{C}_n$ with the bar construction on $\Omega \mathscr{A}'_{k+1}$ to the bar construction on $\Omega \mathscr{F}'_{n+k+1}$. On passing to the geometric realizations of the simplicial spaces, one obtains a map

$$(B\Omega \mathscr{A}'_{k+1}) \wedge \mathscr{C}_n \to B\Omega \mathscr{F}'_{n+k+1} \tag{2.3}$$

that preserves the bar filtration as required.

For connected spaces it is well-known that there is a natural homotopy equivalence $B\Omega \to Id$ (e.g., [**1**], 6.16; [**9**], 2.3, 2.8), so $B\Omega \mathscr{A}'_{k+1}$ and $B\Omega \mathscr{F}'_{n+k+1}$ are equivalent to $\mathscr{A}'_{k+1}$ and $\mathscr{F}'_{n+k+1}$ respectively. Further, the map (2.3) corresponds to the map (2.1) under this equivalence.

The fact that the filtration quotients have the homotopy types indicated in Theorem 1.6 (ii) is also well-known.

This completes the proof of Theorem 1.6. Theorem 1.7 follows immediately.

There are two machines for producing pairings of spectra, one due to May ([**3**] IX) and one due to Segal ([**8**], §5). In the spectra built by these machines, the spaces $\mathscr{A}_k$ and $\mathscr{F}_{k+n}$ carry E_∞- space or Γ space structures that allow one to form bar constructions $B\mathscr{A}_k$ and $B\mathscr{F}_{k+n}$ after converting the E_∞-structure to a monoid structure in one of the standard ways or by directly using the Γ-space structure. For the machines of May and Segal there are "multiplicative" homotopy equivalences of monoids between $\mathscr{A}_k$, $\mathscr{F}_{k+n}$ and the monoids of Moore loops $\Omega \mathscr{A}_{k+1}$, $\Omega \mathscr{F}_{k+n+1}$ respectively ([**9**], 2.3, 2.8, 3.7, 3.10). It follows that there are maps

$$B\mathscr{A}_k \to B\Omega \mathscr{A}_{k+1} \qquad BF_{k+n} \to B\Omega \mathscr{F}_{k+n+1}$$

that induce homotopy equivalences on all filtrations and filtration quotients of the bar filtration. Thus, Theorems 1.6 and 1.7 apply also to the bar construction on the machine-provided monoid structure as well as to the Moore loop structure on $\mathscr{A}_k \simeq \Omega\mathscr{A}_{k+1}$. In particular, it is legitimate to apply the results to the machine-built Eilenberg-MacLane spectra in [**5**].

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