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Topics in Homotopy Theory and Cohomology Theory

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Unstable Cohomology Operations

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Let $E^*(-)$ be a multiplicative generalized cohomology theory. This is represented by a spectrum E which can be represented as an Ω-spectrum

$$E_* = \{\underline{E}_k\}_k, \quad \Omega\underline{E}_{k+1} \simeq \underline{E}_k .$$

Then we have

$$E^k X \simeq [X, \underline{E}_k],$$

or

$$E^* X = [X, \underline{E}_*].$$

We are interested in the unstable $E^*(-)$ cohomology operations, or the natural transformations

$$E^k X \longrightarrow E^n X.$$

We have that

$$\begin{array}{ccc} E^k X & \xrightarrow{\text{n.t.}} & E^n X \\ \simeq\updownarrow & & \updownarrow\simeq \\ [X, \underline{E}_k] & \xrightarrow{\text{n.t.}} & [X, \underline{E}_n], \end{array}$$

and so the natural transformations are given by

$$[\underline{E}_k, \underline{E}_n] \simeq E^n \underline{E}_k.$$

Consequently, $E^*\underline{E}_*$ is of interest. However, we will restrict our attention to additive operations, i.e. those r where

$$r(x+y) = r(x) + r(y).$$

To do this we will assume that

$$E^*(\underline{E}_k \times \underline{E}_k) \simeq E^*\underline{E}_k \hat{\otimes}_{E^*} E^*\underline{E}_k .$$

Then the additive operations are just the primitives:

$$r \in PE^*\underline{E}_* \quad \text{if} \quad r \to r \hat{\otimes} 1 + 1 \hat{\otimes} r .$$

We can rigorously make $PE^*\underline{E}_*$ into a ring such that for any space X, E^*X is an "unstable E^*E module" over the ring $PE^*\underline{E}_*$. The details will appear elsewhere but the concept is fairly clear. In the case of E^*X we have a map

$$PE^n\underline{E}_k \otimes E^kX \longrightarrow E^nX$$

with a number of obvious compatibility conditions; among them the commuting of the diagram:

$$\begin{array}{ccc} PE^i\underline{E}_n \otimes PE^n\underline{E}_k \otimes E^kX & \longrightarrow & PE^i\underline{E}_n \otimes E^nX \\ \downarrow & & \downarrow \\ PE^i\underline{E}_k \otimes E^kX & \longrightarrow & E^iX , \end{array}$$

where the "ring" structure on $PE^*\underline{E}_*$ is clearly going to be given by composition of maps:

$$\begin{array}{ccc} PE^i\underline{E}_n \otimes PE^n\underline{E}_k & \longrightarrow & PE^i\underline{E}_k \\ \cap & & \cap \\ [\underline{E}_n, \underline{E}_i] \otimes [\underline{E}_k, \underline{E}_n] & \longrightarrow & [\underline{E}_k, \underline{E}_i]. \end{array}$$

There is a map, cohomology suspension;

$$E^{k-n}E \longrightarrow PE^k\underline{E}_n$$

from the stable operations to the unstable operations. This is just given by restricting a stable operation to classes of degree n. An example of the potential usefulness is the nondesuspension problem.

If X has a desuspension $\Sigma^{-1}X$, then by the suspension isomorphism and the fact that stable operations commute with suspension, we have a stable E^*E module structure on $\tilde{E}^*(\Sigma^{-1}X)$. However, if $\Sigma^{-1}X$ exists, it must also have an unstable module structure compatible with the stable structure, i.e. we must be able to complete the diagram:

$$\begin{array}{ccc} E^{k-n}E \otimes \tilde{E}^n\Sigma^{-1}X & \longrightarrow & \tilde{E}^k\Sigma^{-1}X \\ \downarrow & \nearrow & \\ PE^k\underline{E}_n \otimes \tilde{E}^n\Sigma^{-1}X & & \end{array}$$

If this cannot be done, then $\Sigma^{-1}X$ does not exist.

We have specific examples for E in mind. In particular we want E to give complex cobordism or Brown-Peterson cohomology. The definition above, however, works for standard mod (p) cohomology as well.

In particularly nice cases,

$$E^*\underline{E}_k \simeq \hom_{E_*}(E_*\underline{E}_k, E_*)$$

and

$$PE^*\underline{E}_k \simeq \hom_{E_*}(QE_*\underline{E}_k, E_*).$$

Both BP and MU satisfy this property. Much more can be said. Hence forth,

$$\text{let} \quad E = MU \text{ or } BP.$$

In these cases

$$E_*(\underline{E}_k \times \underline{E}_k) \simeq E_*\underline{E}_k \hat{\otimes}_{E_*} E_*\underline{E}_k$$

and the diagonal map

$$\underline{E}_k \to \underline{E}_k \times \underline{E}_k$$

turns $E_*\underline{E}_k$ into a coalgebra.

Because $\underline{E}_k$ is a homotopy commutative H-space, $E_*\underline{E}_k$ is a commutative Hopf algebra, with conjugation, over E_*; or, in other words, an abelian group object in the category of coalgebras over E_*. Even more structure exists; since E_* is a ring spectrum we have maps

$$\underline{E}_k \wedge \underline{E}_n \longrightarrow \underline{E}_{k+n}$$

giving us a product

$$\circ : E_*\underline{E}_k \otimes_{E_*} E_*\underline{E}_n \longrightarrow E_*\underline{E}_{k+n},$$

and turning $E_*\underline{E}_* = \{E_*\underline{E}_k\}_k$ into a graded ring object over the category of coalgebras over E_*.

This goes as: E^*X is a graded ring, so $\underline{E}_*$ is a graded ring object in the homotopy category, so $E_*\underline{E}_*$ is a graded ring object in the category of E_*-coalgebras.

The distributivity in this "ring", known as a "Hopf ring", uses the coproduct: let

$$x \longrightarrow \Sigma x' \times x'',$$

then

$$x \circ (y * z) = \Sigma \pm (x' \circ y) * (x'' \circ z)$$

where $*$ is the Hopf algebra product, or "addition" in our "ring".

$$E^*CP^\infty \simeq E^*[[x]] \quad \text{for} \quad x \in E^2CP^\infty.$$

Dual to x^i we have $\beta_i \in E_2CP^\infty$.

We obtain a formal group law over E_* by applying $E^*(-)$ to the usual map

$$CP^\infty \times CP^\infty \to CP^\infty.$$

Then

$$x \to \sum_{i,j} a_{ij} x_1^i \otimes x_2^j = F(x_1, x_2).$$

Define

$$x +_F y = F(x,y) = \sum_{i,j} a_{ij} x^i y^j.$$

We define a few elements in $E_*\underline{E}_*$.

Using

$$x \in E^2 CP^\infty = [CP^\infty, \underline{E}_2]$$

we define

$$b_i \equiv x_*(\beta_i) \in E_{2i}\underline{E}_2.$$

Also for

$$a \in E^k = [pt., \underline{E}_k]$$

we have

$$[a] \equiv a_*(1) \in E_0\underline{E}_k.$$

we define

$$x +_{[F]} y = \mathop{\ast}_{i,j} [a_{ij}] \circ x^{\circ i} \circ y^{\circ j}.$$

In "The Hopf ring for complex cobordism", Journal of Pure and Applied Algebra, 1977, Ravenel and Wilson prove the following about MU and BP. Let $b(s) = \sum_{i \geq 0} b_i s^i$.

<u>Theorem.</u> In $E_*\underline{E}_*[[s,t]]$, $E = MU$ or BP,

$$b(s +_F t) = b(s) +_{[F]} b(t).$$

The Hopf ring E_*E_{2*} is generated over E_* by the b's and $[E^*]$, and the only relations come from above. To obtain $E_*\underline{E}_*$ just add $e_1 \in E_1E_1$ and $e_1 \circ e_1 = b_1$. □

These formulas, by duality, give all information about unstable MU and BP operations. However, there is another way to look at these unstable operations. For $n > 0$ we have the rational isomorphisms

$$E^*E_Q \simeq PE^*\underline{E}_{nQ}.$$

Since there is no torsion anywhere we have

$$\begin{array}{ccccc} E^{*+n}E & \subset & E^{*+n}E_Q & \simeq & \hom_{E_*}(E_{*+n}E,\ E_{*Q}) \\ \downarrow & & \downarrow\simeq & & \downarrow\simeq \\ PE^*\underline{E}_n & \subset & PE^*\underline{E}_{nQ} & \simeq & \hom_{E_*}(QE_*\underline{E}_n,\ E_{*Q}) \end{array}$$

and we can represent an unstable operation by a rational stable operation, However, we have the following surprising result:

<u>Theorem</u>. For $E = MU$ or BP, the coker in

$$0 \to E^{*-n}E \to PE^*\underline{E}_n \to \text{coker} \to 0$$

has no torsion. □

This may seem like a contradiction, but because of completion problems it is not. We have that

$$E^*E \simeq E^* \hat{\otimes} S$$

where S has only nonnegative degrees. E^* has only non positive degrees. When we say "rationally" we mean

$$E^*E_Q \simeq E^*_Q \hat{\otimes} S,$$

not tensor product with Q. In this completed tensor product, an element which is non trivial in the coker is an infinite sum

$$\Sigma a_i \hat{\otimes} s_i\ ,\quad a_i \in E^*_Q\ ,\ s_i \in S,$$

with the denominators of the a_i going to infinity as i does.

A canditate for an unstable operation can be checked now. If we are given an element of $E^*\underline{E}_Q$ we can evaluate it in

$$\hom_{E_*}(E_*\underline{E}_n, E_{*Q})$$

and if we find that all of our values are really in

$$E_* \subset E_{*Q},$$

then we have a legitimate element of

$$PE^*\underline{E}_n .$$

It is at this stage that the detailed knowledge of $E_*\underline{E}_*$ developed in "The Hopf ring for complex cobordism" is useful.

An example of an unstable operation found in this way is the Adams operation ψ^k. These have been studied by several authors rationally, however we can obtain the following by use of the above technique.

Theorem. For $E = MU$ or BP, the rational operations $k^i\psi^k$ actually lie in

$$PE^{2i}\underline{E}_{2i} \text{ and } PE^{2i+1}\underline{E}_{2i+1}, \text{ all } i. \qquad \square$$

In order to prove this type of result, techniques for evaluating

$$E_*(r) : E_*\underline{E}_k \to E_*\underline{E}_n \quad \text{for} \quad r : \underline{E}_k \to \underline{E}_n$$

are necessary.

The details of these techniques, the last two theorems, and the rigorous definition of general unstable operations will appear elsewhere.

This paper represents a portion of the lectures I gave at a conference at the Research Institute for Mathematical Sciences at Kyoto University in October 1980. I would like to thank the participants and organizers for a most enjoyable conference.