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TOWARDS BP_*X

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Let $MU = \{MU(n)\}_n$ be the Thom spectrum for the unitary group. We have a generalized homology theory [A]

$$MU_*X \equiv \pi_*(MU \wedge X) ,$$

with $MU_* \simeq Z[x_2, x_4, \ldots]$. When localized at p , $MU_*X_{(p)}$ is determined by another generalized homology theory, BP_*X , called Brown-Peterson homology. It has representing spectrum BP . We can define generalized cohomology theories

$$MU^*X \equiv \{X, MU\}^*$$

$$BP^*X \equiv \{X, BP\}^*$$

and again, $MU^*X_{(p)}$ is determined by BP^*X . The coefficient ring, BP_* , is

$$BP_* \simeq Z_{(p)}[v_1, v_2, \ldots]$$
$$|v_n| = 2(p^n-1) .$$

During the 1970's there were many applications of BP in algebraic topology. In particular, applications to stable homotopy came with a deeper understanding of stable operations. For example, see [MRW]. However, the ability to compute BP_*X for commonly occurring X was missing. I expect this situation to change dramatically during the 1980's; and things are already off to a good start. Before I discuss recent developments I want to review what is known.

The BP homology of spaces with few cells can be computed and used to great advantage in stable homotopy. Typical examples are the $V(n)$ spaces which exist in limited quantities for small n and certain primes. The defining property of a $V(n)$ is that the mod (p) cohomology

$$H^*V(n) \simeq E_{n+1} \equiv E(Q_0, Q_1, \ldots , Q_n) ,$$

where the Q_i are the Milnor Bocksteins [Mil]. Equivalently [S]

$$BP_*V(n) \simeq BP_*/I_{n+1} ,$$

$I_{n+1} \equiv (p, v_1, \ldots , v_n)$. But note, for future use, that

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$$BP_*V(n-1) \simeq BP_*/I_n$$

while

$$BP^*V(n-1) \simeq \Sigma^{2(p^n-1)/(p-1)-n}BP^*/I_n \ .$$

At any rate, the BP homology of spaces with very few cells can usually be computed, and frequently used for applications.

Next, there is a collection of artificial spaces constructed to demonstrate particular properties of BP_*X, with very little additional interest.

There are also some scattered, unsystematic results around.

The most accessible spaces are those with no torsion in homology. The Atiyah-Hirzebruch spectral sequence collapses and we automatically have the BP_* module structure (it's free). So in a very real sense these examples are trivial. However, much more can be done. For example, the algebra structure of $H^*(X, BP^*)$ does not give the algebra structure of BP^*X because of extension problems. These are very important extension problems and it is necessary to understand them completely in order to have such basics as the Landweber-Novikov algebra

$$MU^*MU$$

and its dual MU_*MU. Furthermore, BP was useless until these extension problems could be solved for BP_*BP; which is what Quillen accomplished with his construction of BP. More recently, for the Ω-spectrum $MU = \underline{MU}_*$, [RW$_1$] studies $MU_*\underline{MU}_*$, where there are two products (but no torsion) to give extension problems. But still, these examples are studies in the subtleties of trivialities.

The first serious example, due to Landweber [L$_1$], has been with us quite a while. He computes

$$MU^*(BZ/(k_1) \times BZ/(k_2) \times \ldots \times BZ/(k_n)) \ .$$

The fibration

$$BZ/(k) \longrightarrow \mathbb{C}P^\infty \xrightarrow{k} \mathbb{C}P^\infty$$

gives a short exact sequence in $MU^*(-)$.

$$MU^*\mathbb{C}P^\infty \simeq MU^*[[x]] \ ,$$

$$x \in MU^2\mathbb{C}P^\infty \ .$$

Let $[k](x) = k^*(x)$. Then

$$MU^*BZ/(k) \simeq MU^*[[x]]/([k](x)) \ ,$$

and Landweber shows

$$MU^*(BZ/(k_1) \times \dots \times BZ/(k_n))$$
$$\simeq MU^*[[x_1, x_2, \dots, x_n]]/([k_1](x_1), \dots, [k_n](x_n)) .$$

He succeeds in proving this by showing there is a Künneth isomorphism for these spaces; a rare phenomenon.

Despite very early interest [CF], homology versions of the above were not successful until 1980 when David Johnson and the author computed

$$BP_*(\times^n BZ/(p)) ,$$

the first example of a BP homology calculation for a sequence of standard spaces with progressively more complicated BP_* module structure. ($n = 1, 2, 3,$ were known.)

The $BZ/(p)$ are basic spaces in algebraic topology and immediate geometric consequences follow. This determines (for p an odd prime)

$$MSO_* \times^n BZ/(p)$$

which gives the bordism classes of oriented manifolds with free $\times^n Z/(p)$ actions.

I have a basic interest in developing the computability of $BP_*(-)$. The best place to begin is with the standard spaces of algebraic topology, and, as above, applications will follow. Other important spaces are BO_n and MO_n , and it is my recent computation of $BP_*(-)$ and $BP^*(-)$ $(p = 2)$ for these spaces that I want to talk about.

Before I discuss the answers, I want to describe the technique for the spaces $\times^n BZ/(p)$, BO_n , and MO_n . We use the Adams spectral sequence

$$E_2^{**} \simeq \mathrm{Ext}_A(H^*(BP\wedge X), Z/(p)) \Longrightarrow \pi_*(BP\wedge X) \simeq BP_*X .$$

H^*BP is the reduced p-th powers, so by a change of rings we have

$$E_2^{**} \simeq \mathrm{Ext}_E(H^*X, Z/(p)) ,$$

where $E = E(Q_0, Q_1, \dots)$, an exterior algebra on the Milnor Bocksteins. A major simplification can be made for the above mentioned spaces. Ext_E is determined by Ext_{E_n} in these cases. This comes out in the computations, but it is a very comforting fact to know in advance, which for BO_n it was, by Ron Ming's work [Min]. Ming goes even further and shows that $\mathrm{Ext}_E^{**}(H^*BO_n, Z/(2))$ is generated over BP_* by Ext^0 , which is unreasonable to hope for, but it helps the properly paranoid believe that the spectral sequence can be made to collapse because the only elements to check are the accessible Ext^0 elements.

So, to compute for these spaces, compute the Ext over E_n , a nice finite exterior algebra with none of the complications of the Steenrod algebra. Then show $E_2 = E_\infty$, and in some sense we are done.

However, collapsing is hard. To do the Johnson-Wilson collapsing for $BP_*\times^n BZ/(p)$ we have to go back to the Ravenel-Wilson computation of the Morava K-theories of Eilenberg-MacLane spaces. In [RW_2] the Morava structure theorem is used to compute $v_n^{-1}BP_*K(Z/(p), n)$. Comparing

$$BP_*\times^n BZ/(p) \to v_n^{-1}BP_*K(Z/(p), n) ,$$

Ravenel-Wilson prove what we call the Conner-Floyd conjecture [CF], i.e. that certain elements are non-zero in $BP_*\times^n BZ/(p)$. Johnson-Wilson show that if $E_2 \neq E_\infty$, these elements must get hit by differentials, so $E_2 = E_\infty$. The hardest part of BO_n is the collapsing. Part of the proof is a comparison of

$$\times^n BZ/(2) \to BO_n .$$

However, the computations for these two are not done in a compatible way, so the proof is very complicated.

I owe a debt to the work of Ron Ming and the successful efforts of the joint work with David Johnson for motivating my attempt at this problem. The collapsing of the spectral sequence is entirely dependent upon the collapsing for $\times^n BZ/(2)$, which again is entirely dependent on the computation of $K(n)_*\underline{K}_*$ in [RW_2].

Collapsing for $BP^*\times^n BZ/(p)$ and BP^*BO_n is trivial. So, we have "computed" BP_*X , BP^*X , $X = \times^n BZ/(p)$, BO_n , MO_n . The cofibration

$$BO_{n-1} \to BO_n \to MO_n$$

gives a short exact sequence in both $BP_*(-)$ and $BP^*(-)$.

The Adams spectral sequence answer is a beginning, (an important beginning), but not really an acceptable answer. What we need is an internal $BP_*(-)$ description of the answer, with the Adams spectral sequence computation just a step in the proof.

For $\times^n BZ/(p)$ we have this. Let $\widetilde{BP}_*BZ/(p) \equiv N$, $\otimes^n_{BP_*} N \equiv N^n$ and L_n the free module on generators in degree $0 < 2i < 2p^n$.

THEOREM (Landweber [L_2])

$$0 \to N\otimes_{BP_*}\widetilde{BP}_*X \to \widetilde{BP}_*(BZ/(p)\wedge X) \to \mathrm{Tor}_1^{BP_*}(N, \widetilde{BP}_*X) \to 0 \qquad \square$$

THEOREM (Johnson-Wilson) There is a filtration on $BP_*\times^n BZ/(p)$ such that the associated graded object is one copy of N^n , and many copies of N^k , $k < n$. $\square$

SKETCH PROOF First we show

THEOREM (Johnson-Wilson)

$$\mathrm{Tor}_1^{BP_*}(N, N^n) \simeq L_n N^n \qquad \square$$

Inductively we use the Künneth short exact sequence and this computation of Tor . We cannot do this without using the additional information that we know the orders of the groups $BP_* \times^n BZ/(p)$ from our Adams spectral sequence computation. This is all we use from it. $\square$

Of course we give an accounting of the number of copies of N^k in the complete statement of the theorem. Our counting can be reproduced by making the unjustified assumption that the Künneth short exact sequence splits; just use the Tor theorem inductively.

Getting a good description for BO_n is still work-in-progress at a very preliminary stage. However, it looks like it will turn out nicely. We have maps

$$BO_{n-2} \times BO_2 \to BO_n$$

and

$$\times^n BZ/(2) \to BO_n \; .$$

THEOREM BP_*BO_n is generated by the images

$$BP_*BO_{n-2} \otimes BP_*BO_2 \to BP_*BO_n \quad \text{and}$$
$$BP_* \times^n BZ/(2) \to BP_*BO_n \qquad \square$$

This explains the proof of the collapsing of the spectral sequence. For the first part of the theorem it is enough to use the "co-Pontrjagin" classes $x_{4i} \in BP_{4i}BO_2$.

We can say more, the symmetric group $\sum_n$ acts on $BP_* \times^n BZ/(2)$ and our map clearly factors (all x , all $\sigma \in \sum_n$)

$$BP_* \times^n BZ/(2) \to BP_* \times^n BZ/(2)/x = \sigma x \to BP_*BO_n \; .$$

However, the second map is not injective because v_n torsion is created, so we should go to

$$\begin{tikzcd}
BP_* \times^n BZ/(2) \arrow[r] \arrow[d, phantom, "\bigcap"] & BP_* \times^n BZ/(2)/x = \sigma x \arrow[r] & BP_*BO_n \arrow[d, phantom, "\bigcap"] \\
v_n^{-1} BP_* \times^n BZ/(2) \arrow[r] & v_n^{-1} BP_* \times^n BZ/(2)/x = \sigma x \arrow[r] & v_n^{-1} BP_*BO_n
\end{tikzcd}$$

Perhaps this last map injects and solves that problem for us.

Let's go to BP^*BO_n , or better, MU^*BO_n . There is no problem at p odd but the description appears more complete using MU . We have the universal bundles and the complexification map

$$\begin{tikzcd}
\zeta_n \otimes \mathbb{C} \arrow[rr] \arrow[dr] \arrow[d] & & (\zeta_n^{\mathbb{C}})^* \arrow[d, dash] \\
BO_n \arrow[rr] & \zeta_n^{\mathbb{C}} \arrow[r] & BU_n
\end{tikzcd} ,$$

where $*$ is the complex conjugate and $(\zeta_n \otimes \mathbb{C})^* \simeq (\zeta_n \otimes \mathbb{C})$. We know $MU^*BU_n \simeq MU^*[[C_1, \ldots , C_n]] \simeq MU^*[[C_1^*, \ldots , C_n^*]]$ where C_k is the Chern class and C_k^* is the Chern class of the conjugate bundle. Thus we have a map.

$$MU^*BO_n \leftarrow MU^*[[C_1, \ldots , C_n]]/(C_k = C_k^*) .$$

THEOREM This map is surjective. □

This explains why it is trivial to get the ASS to collapse.

CONJECTURE This map is an isomorphism. □

It appears that eventually a good description of the answer will be available, using the Adams spectral sequence in an essential way in the proof. Both $BP^*\times^n BZ/(p)$ and BP^*BO_n have reasonable answers coming from $\times^n \mathbb{C}P^\infty$ and BU_n. The answers for $BP_*\times^n BZ/(p)$ and BP_*BO_n are not so easily comprehended. There is a relationship, "perverted duality," between these cohomology and homology groups.

H^*X, for $\times^n BZ/(p)$, BO_n, and MO_n, can be adequately described as an E module by: There are E generators

$$F = F_0 \oplus F_1 \oplus \ldots \oplus F_n$$

with

$$H^*X \simeq E_0F_0 \oplus E_1F_1 \oplus \ldots \oplus E_nF_n .$$

For any E module with such a representation we can easily compute the graded object associated with a filtration of

$$\mathrm{Ext}_E(H^*X, Z/(p)) .$$

We just get a BP_*/I_k for each basis element F_k. Of course this type of cohomology decomposition does not occur in general and it is partially responsible for the computability of BP_*X for these X.

We now have a duality analogous to that for $V(n-1)$ discussed earlier. For each E_k that occurs (for all F_k) we have a BP_*/I_k and a $\Sigma^{2(p^k-1)/(p-1)-k}BP^*/I_k$ in our associated graded computations for Ext_E. ($\mathrm{Ext}_E(Z/(p), H^*X) \Longrightarrow BP^*X$.) So our "duality" gives a shift by different degrees, depending on k, and our reasonable BP^*X demands a radically different BP_*X.

To conclude, we state the E module structure for H^*BO_n as above by giving the F_k. We have $H^*BO_n \subset H^*\times^n BZ/(2)$ as the symmetric functions on $t_1, \ldots , t_n$.

THEOREM A basis for F_k, $0 \le k \le n$, as above, is given by all

$$\sum_{sym} t_1^{2i_1+1} t_2^{2i_2+1} \cdots t_k^{2i_k+1} t_{k+1}^{2j_1} t_{k+2}^{2j_2} \cdots t_{k+q}^{2j_q} , \ k + q \le n ,$$

with $0 \le i_1 \le i_2 \le \ldots \le i_k$, $0 \le j_1 \le j_2 \le \ldots \le j_q$, where, if the number of j's equal to j_u is odd, then there is some $s > 0$, such that

$$2i_s + 2^s < 2j_u < 2i_s + 2^{s+1} .$$ □

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