THE COMPLEX COBORDISM OF BO_n

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Introduction

A surprising simplicity appears in the complex cobordism of the classifying space of the n-th orthogonal group, MU^*BO_n. There is a close relationship to the complex cobordism of the classifying space of the n-th unitary group, MU^*BU_n. The latter is the power series ring on the Conner–Floyd Chern classes [**8**]:

$$MU^*BU_n \simeq MU^*[[C_1, \ldots, C_n]], \qquad C_k \in MU^{2k}BU_n .$$

The complexification, $\xi_n \otimes \mathbb{C}$, of the universal n-dimensional real bundle ξ_n over BO_n, induces a map of BO_n to BU_n covered by a map to the universal n-dimensional complex bundle $\xi_n^{\mathbb{C}}$:

$$\begin{array}{ccc} \xi_n \otimes \mathbb{C} & \longrightarrow & \xi_n^{\mathbb{C}} \\ \downarrow & & \downarrow \\ BO_n & \longrightarrow & BU_n . \end{array}$$

This map gives us the Conner–Floyd Chern classes of $\xi_n \otimes \mathbb{C}$. The bundle $\xi_n \otimes \mathbb{C}$ is isomorphic to its own complex conjugate. Consequently, its Conner–Floyd Chern class C_k must be equal to the image (under this map) of the k-th Conner–Floyd Chern class C_k^* of the complex conjugate of the universal bundle $(\xi_n^{\mathbb{C}})^*$. Our result is as follows.

THEOREM 1.

$$MU^*BO_n \simeq MU^*[[C_1, \ldots, C_n]]/(C_1 - C_1^*, \ldots, C_n - C_n^*) .$$

The elegance of the statement of the theorem attempts to compensate for its depressing uselessness. There are no new, interesting, complex cobordism characteristic classes for real bundles; but only the well-studied Conner–Floyd Chern classes. The motivation for attacking MU^*BO_n was the remarkable success L. Astey had applying complex cobordism to the generalized vector field problem [**4**, **5**]. Intuitively, we felt that if complex cobordism held so much information for this problem, it should show up best on the classifying space level. Theorem 1 shows otherwise.

The answer, however, does suggest generalization: many more theories than $MU^*(\cdot)$ must give this answer, and they should all come with a general nonsense proof. The immediate thought, that any complex orientable theory will do, fails. The result is similar in flavor to Landweber's results about the complex cobordism of $\times^k BZ/(n_k)$, which also has the simplest possible answer [**12**].

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The standard long exact sequence becomes short exact:

$$0 \longleftarrow MU^*BO_{n-1} \longleftarrow MU^*BO_n \longleftarrow \widetilde{MU}^*MO_n \longleftarrow 0,$$

giving the following result.

THEOREM 2. $\widetilde{MU}^*MO_n \simeq (C_n) \subset MU^*BO_n$, *the ideal generated by* C_n.

The problem is all 2-primary since, when localized at an odd prime,

$$MU^*_{(p)}BO_n \simeq MU^*_{(p)}[[C_2, C_4, \ldots, C_{2k}, \ldots]], \quad 2k \leqslant n,$$

a well-known, obvious result. Consequently, we concentrate on the case when $p = 2$. We use the equivalent Brown–Peterson cohomology for $p = 2$, and prove Theorem 1 for BP^*BO_n [**3**, **6**, **17**, **19**]. Except for the fact that it works, the proof has no redeeming value. We compute the Adams spectral sequence for BP^*BO_n, which, by a change of rings, becomes

$$E_2 \simeq \mathrm{Ext}_E(\mathbb{Z}/2, H^*(BO_n, \mathbb{Z}/2)) \Rightarrow BP^*BO_n,$$

where $E = E(Q_0, Q_1, \ldots)$, the exterior algebra on the Milnor primitives. The spectral sequence collapses and the map from BP^*BU_n is obviously surjective. The relations $C_k = C_k^*$ must hold. All that remains is to show that the right side of Theorem 1 injects. The left side, BP^*BO_n, is computed explicitly, but inelegantly. The right side is very elegant, but with no detail. The right-hand side is filtered and studied. We show that it is no bigger than BP^*BO_n. This completes the proof.

Along the way we compute BP_*BO_n (equivalently MU_*BO_n) and BP_*MO_n using the Adams spectral sequence. An elegant description of BP_*BO_n still escapes us, but we do know that BP_*BO_n is generated by the images of

$$BP_*(\times^n RP^\infty) \longrightarrow BP_*BO_n$$

and

$$BP_*BO_{n-2} \otimes BP_*BO_2 \longrightarrow BP_*BO_n$$

where it is enough to use the $4k$-dimensional torsion free generators of BP_*BO_2. A nice description of BP_*BO_n should emerge from these facts. We refrain from proving these results in this paper. They are very complicated and, at present, lead nowhere. They were originally necessary for showing the collapse of the Adams spectral sequence for BP_*BO_n. That argument has now been replaced by Ravenel's trick for deducing this collapse from BP^*BO_n.

Recall ($p = 2$)

$$BP_* \simeq \mathbb{Z}_{(2)}[v_1, v_2, \ldots], \qquad I_n = (2, v_1, \ldots, v_{n-1}).$$

THEOREM 3. *There is a* BP_* *module filtration on* $\widetilde{BP}_*BO_n$ *such that the associated graded* BP_* *module is generated by the reduction of elements in* $\widetilde{BP}_*BO_n$ *which inject to* $\tilde{H}_*(BO_n, \mathbb{Z}/2)$. *The associated graded module has a* BP_*/I_k *in dimension*

$$k + 2\sum_{t=1}^{k} i_t + 2\sum_{v=1}^{q} j_v,$$

with $0 \leqslant i_1 \leqslant \ldots \leqslant i_k$ *and* $0 < j_1 \leqslant \ldots \leqslant j_q$, $k+q \leqslant n$ *where, if the number of* j *equal to* j_u *is odd, then there is some* s *such that*

$$2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}.$$

Despite its satisfactory complexity, this result does no better than Theorem 1 in a direct approach to the generalized vector field problem.

This paper owes a major debt to two sources. The approach, techniques, and results of the recent paper [11] with David Johnson are all essential to the computation done here. This paper is an obvious continuation of our joint work and had we been close enough to communicate during the early stages it most likely would have become another joint paper. Ron Ming initiated this approach to $BP_* BO_n$ in his paper [**15**]. He shows that BO_n is a particularly good candidate for computation by showing that the E_2 term of the Adams spectral sequence for $BP_* BO_n$ is determined by

$$\mathrm{Ext}_{E_n}(H^*(BO_n, \mathbb{Z}/2), \mathbb{Z}/2)$$

where $E_n = E(Q_0, \ldots, Q_{n-1})$. This hints at a possible projective dimension for $BP_* BO_n$ of n (it is), which severely restricts its complexity. Furthermore, he shows that this Ext is generated by Ext^0 as a BP_* module. This makes it seem likely that one could either prove that the spectral sequence collapses or compute its differentials.

Some philosophizing about this type of result can be found in [**20**].

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In Section 1 we review the facts we need and set up our notation. The Adams spectral sequence computation is carried out in Section 2 and $BP^* BU_n/(C_k - C_k^*)$ is analysed in Section 3.

1. *Preliminaries*

Basic references for Brown–Peterson homology are [**3, 6, 17, 19**]. We are concerned with BP for $p = 2$.

We use the $p = 2$ Adams spectral sequence. Let $H^* X$ be the mod 2 cohomology of X. Let $\mathbb{Z}_2$ be the 2-adic integers and let A be the mod 2 Steenrod algebra. The Adams spectral sequence [1]

1.1 $$E_2^{**} \simeq \mathrm{Ext}_A^{**}(H^* X, H^* Y) \Rightarrow \{Y, X\}_* \otimes \mathbb{Z}_2$$

can be used to compute

1.2 $$BP^* Y \simeq \{Y, BP\}_{-*} \quad \text{and} \quad BP_* X \simeq \{S^0, BP \wedge X\}_*.$$

Let

1.3 $$E \equiv E(Q_0, Q_1, \ldots)$$

be the exterior algebra on the Milnor primitives [13]. Now E is a normal subalgebra of A and we have

1.4 $$E \hookrightarrow A \longrightarrow A//E \simeq H^*\mathrm{BP}\,.$$

By the Cartan–Eilenberg change of rings spectral sequence we can replace

1.5 $$\mathrm{Ext}_A^{**}(H^*(\mathrm{BP} \wedge X), H^*Y) \quad \text{with} \quad \mathrm{Ext}_E^{**}(H^*X, H^*Y)\,.$$

(See [7] or, for this particular case, [15].) The forms of the Adams spectral sequence we use are

1.6 $$\mathrm{Ext}_E^{**}(H^*X, \mathbb{Z}/2) \Rightarrow \mathrm{BP}_*X$$

and

1.7 $$\mathrm{Ext}_E^{**}(\mathbb{Z}/2, H^*Y) \Rightarrow \mathrm{BP}^{-*}Y\,.$$

Recall that

1.8 $$\mathrm{BP}_* \simeq \mathbb{Z}_{(2)}[v_1, v_2, \ldots], \quad |v_n| = 2(2^n - 1) \quad \text{and} \quad I_n = (2, v_1, \ldots, v_{n-1})\,.$$

We have $\mathrm{BP}_* \Leftarrow \mathrm{Ext}_E^{**}(\mathbb{Z}/2, \mathbb{Z}/2) \simeq \mathbb{Z}/2[v_0, v_1, \ldots] \equiv \mathrm{BP}'_*$ where

1.9 $$v_n \in \mathrm{Ext}^{1, 2^{n+1}-1}(\mathbb{Z}/2, \mathbb{Z}/2)$$

and $I'_n \equiv (v_0, v_1, \ldots, v_{n-1})$. Let

1.10 $$E_k \equiv E(Q_0, Q_1, \ldots, Q_{k-1})\,.$$

Considering this as a quotient of E, we have

1.11 $$\begin{cases} \mathrm{Ext}_E^{**}(E_k, \mathbb{Z}/2) \simeq \mathrm{BP}'_*/I'_k\,, \\ \mathrm{Ext}_E^{**}(\mathbb{Z}/2, E_k) \simeq \sum^{k+2-2^{k+1}} \mathrm{BP}'_*/I'_k\,. \end{cases}$$

We seldom compute $\mathrm{Ext}_R(M, N)$ directly, but usually use the spectral sequence associated with a filtration. Let $\bar{M}$ and $\bar{N}$ be the associated graded R modules from filtrations of M and N respectively. Then we have

1.12 $$E_1^{**} \simeq \mathrm{Ext}_R^*(\bar{M}, N) \Rightarrow \mathrm{Ext}_R^*(M, N)$$

and

1.13 $$E_1^{**} \simeq \mathrm{Ext}_R^*(M, \bar{N}) \Rightarrow \mathrm{Ext}_R^*(M, N)\,,$$

where differentials raise cohomological degree by one.

To do Ravenel's trick we need the duality spectral sequence [**2**, **9**, **10**]

1.14 $$E_2^{s,t} \simeq \mathrm{Ext}^{s,t}_{\mathrm{BP}_*}(\mathrm{BP}_*X, \mathrm{BP}^*) \Rightarrow \mathrm{BP}^*X$$

and, (see [**11**, 6.5])

1.15 $$\mathrm{Ext}^{*,*}_{\mathrm{BP}_*}(\mathrm{BP}_*/I_k, \mathrm{BP}^*) \simeq \mathrm{Ext}^{k,*}_{\mathrm{BP}_*}(\mathrm{BP}_*/I_k, \mathrm{BP}^*) \simeq \textstyle\sum^s \mathrm{BP}^*/I_k$$

where $s = \sum_{0 \leqslant i < k} 2^{i+1} - 2 = 2^{k+1} - 2 - 2k$.

We review what we need about the mod 2 cohomology of various spaces. There are many references, for example [**14**, **18**]. Let $\mathbb{Z}/2[x_1, \dots, x_n]$ be the polynomial algebra on the x_i of degree one. The mod 2 symmetric functions (those invariant under all permutations of the x_i) form a subring which is a polynomial algebra on the elementary symmetric functions σ_i, $1 \leqslant i \leqslant n$, of degree i $(1 + \sigma_1 + \dots + \sigma_n = (1 + x_i) \dots (1 + x_n))$. We have

1.16 $$\begin{array}{ccc} \mathbb{Z}/2[x_1, \dots, x_n] & \supset & \mathbb{Z}/2[\sigma_1, \dots, \sigma_n] \\ \wr\wr & & \wr\wr \\ H^* \times^n P^\infty & \supset & H^*\mathrm{BO}_n, \end{array}$$

where P^∞ is the infinite dimensional real projective space, and we identify the i-th Stiefel–Whitney class $w_i \in H^i\mathrm{BO}_n$ with the symmetric function σ_i. We also use the fact that the Chern classes are the elementary symmetric functions on the 2-dimensional classes y_i

1.17 $$\begin{array}{ccc} \mathbb{Z}/2[y_1, \dots, y_n] & \supset & \mathbb{Z}/2[c_1, \dots, c_n] \\ \wr\wr & & \wr\wr \\ H^* \times^n \mathbb{C}P^\infty & \supset & H^*\mathrm{BU}_n \\ \cap & & \cap \\ H^* \times^n P^\infty & \supset & H^*\mathrm{BO}_n, \end{array}$$

where the lower inclusion takes y_i to x_i^2.

The E module structure is determined by

1.18 $$Q_k(xy) = Q_k(x)y + xQ_k(y), \qquad |Q_k| = 2^{k+1} - 1, \quad Q_k x_i = x_i^{2^{k+1}}.$$

Two monomials in the x_i are equivalent if some permutation of the x_i takes one to the other. Let $I = (i_1, \dots, i_n)$. Let $x^I = x_1^{i_1} \dots x_n^{i_n}$ and define

1.19 $$s_I = \sum x_1^{i_1} \dots x_n^{i_n} = \sum x^I$$

to be the symmetric function obtained by summing all monomials equivalent to $x_1^{i_1} \dots x_n^{i_n}$, for example

1.20 $$\sigma_k = \sum x_1 \dots x_k .$$

We get a basis for all symmetric functions by using all I with

1.21 $$i_1 \geqslant i_2 \geqslant \dots \geqslant i_n .$$

We order the monomials x^I by $x^I < x^J$ if $i_n < j_n$, or if $i_{k+1} = j_{k+1}, \dots, i_n = j_n$ and $i_k < j_k$. In a sum of monomials we call the smallest monomial the *lead term*. In particular, a basis element of the symmetric functions

1.22 $$s_I = \sum x^I, \qquad i_1 \geqslant i_2 \geqslant \dots \geqslant i_n$$

is determined by its lead term, x^I. Note that if we have two symmetric functions f and g, then

1.23 $$\text{lead term } (fg) = \text{lead term } (f) \text{ lead term } (g) .$$

We turn now to the $p = 2$, Conner–Floyd Chern classes for Brown Peterson Cohomology [**8**, **18**, **3**]. We have

$$\mathrm{BP}^*\mathbb{C}P^\infty \simeq \mathrm{BP}^*[[T]], \qquad T \in \mathrm{BP}^2\mathbb{C}P^\infty$$

1.24 $$\mathrm{BP}^*X^n\mathbb{C}P^\infty \simeq \mathrm{BP}^*[[T_1, T_2, \dots, T_n]]$$

$$\cup \qquad\qquad \cup$$

$$\mathrm{BP}^*\mathrm{BU}_n \simeq \mathrm{BP}^*[[C_1, \dots, C_n]]$$

where C_k is represented by the k-th elementary symmetric function on the T.

The complex conjugate on the one dimensional universal bundle over $\mathrm{BU}_1 \simeq \mathbb{C}P^\infty$ is induced by the map

1.25 $$\mathbb{C}P^\infty \xrightarrow{-1} \mathbb{C}P^\infty .$$

This map takes T to

1.26 $$\iota(T) = \sum_{i \geqslant 0} a_i T^{i+1} .$$

We have that the Chern class is

1.27 $$C_k = \sum T_1 T_2 \dots T_k ,$$

so that the Chern class of the complex conjugate bundle is

1.28 $$C_k^* = \sum \iota(T_1)\iota(T_2) \dots \iota(T_k) .$$

A little formal group manipulation suffices to prove (where $I = (2, v_1, \ldots)$) that

$$1.29 \qquad \begin{cases} a_0 = -1, & \\ a_i \in I^2 & i \neq 2^n - 1,\ n \geq 0, \\ a_i \equiv v_n \bmod I^2 & i = 2^n - 1,\ n > 0; \end{cases}$$

but rather than present the proof here we refer the reader to [**16**].

THEOREM 1.30. *Let $p = 2$, and consider $\mathrm{BP}^*\mathrm{BU} \subset \mathrm{BP}^* \times {}^\infty\mathbb{C}P^\infty$, then*

$$C_k^* \equiv (-1)^k C_k + \sum_{i>0} v_i s_{2^i, \underbrace{1, 1, \ldots, 1}_{k-1}} \bmod I^2.$$

Setting $0 = C_k - C_k^*$, $k > 0$ we have relations (one for each $k > 0$)

$$e_k: \qquad 0 = \sum_{i \geq 0} v_i s_{2^i - 1 + k} \bmod I^2, \qquad \text{where } v_0 = 2.$$

In the ideal $(C_n) \subset \mathrm{BP}^*[[C_1, \ldots, C_n]]/(C_1 - C_1^*, \ldots, C_n - C_n^*)$, we have the additional relation

$$e_0: \qquad 0 = 1 - (-1)^n + \sum_{i>0} v_i s_{2^i - 1} \bmod I^2.$$

Proof. The first formula follows easily from 1.28 and 1.29. The relation e_0 follows by dividing the $k = n$ case by C_n. Let e_k' be the relation obtained from the first formula by setting $0 = C_k - C_k^*$. Recall that we are working modulo $I^2 = (4, 2v_i, v_i v_j)$. We prove e_k by induction on k. We have e_0 already. Observe that e_1' is e_1. For $0 < i < k$,

$$0 = C_i e_{k-i} = C_i \sum_{j \geq 0} v_j s_{2^j - 1 + k - i} = \sum_{j \geq 0} v_j \left(s_{2^j + k - i, \underbrace{1, \ldots, 1}_{i-1}} + s_{2^j - 1 + k - i, \underbrace{1, \ldots, 1}_{i}} \right),$$

except when $j = 0$ and $i = k - 1$ when the last term has a (k) in front. Keeping this in mind, we compute

$$\begin{aligned} 0 &= e_k' + \sum_{i=1}^{k-1} C_i e_{k-i} \\ &= C_k - (-1)^k C_k + \sum_{j>0} v_j s_{2^j, \underbrace{1, \ldots, 1}_{k-1}} + \sum_{i=1}^{k-2} \sum_{j \geq 0} v_j \left(s_{2^j + k - i, \underbrace{1, \ldots, 1}_{i-1}} + s_{2^j - 1 + k - i, \underbrace{1, \ldots, 1}_{i}} \right) \\ &\quad + \sum_{j>0} v_j \left(s_{2^j + 1, \underbrace{1, \ldots, 1}_{k-2}} + s_{2^j, \underbrace{1, \ldots, 1}_{k-1}} \right) + v_0 \left(s_{2, \underbrace{1, \ldots, 1}_{k-2}} + (k) s_{1, \underbrace{1, \ldots, 1}_{k-1}} \right) \\ &= C_k - (-1)^k C_k + \sum_{j \geq 0} v_j s_{2^j + k - 1} + v_0 (k) C_k = \sum_{j \geq 0} v_j s_{2^j - 1 + k} = e_k. \end{aligned}$$

We thank Manuel Moreira for some crucial observations in the finding of the relations e_k.

REMARK 1.31. For odd primes, $\iota(T) = -T$, and we get $C_k^* = (-1)^k C_k$.

2. *The Adams spectral sequence*

All of our notation and terminology is defined in Section 1. We begin with the E module structure of H^*BO_n.

THEOREM 2.1. *As vector spaces over* $\mathbb{Z}/2$,

$$H^*BO_n \simeq \bigoplus_{i=0}^{n} E_i G_i,$$

where the E module generators are given by $\bigoplus_{i=0}^{n} G_i$ *and a basis for* G_k *is given by all symmetric functions*

$$\sum x_1^{2i_1+1} x_2^{2i_2+1} \dots x_k^{2i_k+1} x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q}, \qquad k+q \leqslant n$$

with $0 \leqslant i_1 \leqslant \dots \leqslant i_k$ *and* $0 < j_1 \leqslant \dots \leqslant j_q$; *and if the number of j equal to* j_u *is odd, then there is some* $s \leqslant k$ *such that*

$$2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}.$$

Proof. We show linear independence of $\bigoplus E_i G_i$. Denote the basis elements of G_k by $s_{I_k J_q}$. For

$$K_k = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{k-1}), \qquad \varepsilon_i = 0, 1,$$

we define

$$Q_{K_k} = Q_0^{\varepsilon_0} Q_1^{\varepsilon_i} \dots Q_{k-1}^{\varepsilon_{k-1}}, \qquad K_k' = (1-\varepsilon_0, \dots, 1-\varepsilon_{k-1}),$$

and

$$(1)_k = (1, \dots, 1) = K_k + K_k'.$$

The length of K_k is $\sum \varepsilon_i$.

LEMMA. (a) *The lead term of* $Q_{(1)_k} s_{I_k J_q}$ *is the lead term of*

$$\sum x_1^{2i_1+2} x_2^{2i_2+4} \dots x_k^{2i_k+2^k} x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q}.$$

(b) *The lead terms for the* $Q_{(1)_k} s_{I_k J_q}$ *are all distinct.*

Proof. Recall from 1.18 that $Q_j x_i = x_i^{2^{j+1}}$. In the computation of

$$Q_{(1)_k} s_{I_k J_q} = Q_0 Q_1 \dots Q_{k-1} (\sum x_1^{2i_1+1} \dots x_k^{2i_k+1} x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q})$$

we have the terms

$$\sum Q_0(x_1^{2i_1+1}) \dots Q_{k-1}(x_k^{2i_k+1}) x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q} = \sum x_1^{2i_1+2} \dots x_k^{2i_k+2^k} x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q}$$

because, if $2i_s+2^s = 2j_u$ for some u and s, then by the definition of $s_{I_kJ_q}$ there are an even number of j equal to j_u and so this term is not zero. For example, $Q_0 \sum x_1x_2^2x_3^2 = \sum x_1^2x_2^2x_3^2$, but $Q_0 \sum x_1x_2^2 = 0$. This gives the smallest possible lead term because $i_1 \leqslant \ldots \leqslant i_k$ and $|Q_0| < |Q_1| < \ldots < |Q_{k-1}|$. Part (b) follows from the one to one correspondence produced soon between the set of $Q_{K_k}s_{I_kJ_q}$ and the symmetric functions. In that proof, our map takes $Q_{(1)_k}s_{I_kJ_q}$ to the symmetric function with the above lead term.

Assume a relation

$$0 = \sum a_{K_iI_iJ_q}Q_{K_i}s_{I_iJ_q}.$$

Find the smallest k with some $a_{K_kI_kJ_q} \neq 0 \in \mathbb{Z}/2$. Apply $Q_{K'_k}$ to the above relation. We know that $Q_{K'_k}Q_{K_k}s_{I_kJ_q} = Q_{(1)_k}s_{I_kJ_q}$ has lead term as above. This term is a symmetric function on even powers only. All such terms must cancel out. We study $Q_{K'_k}Q_{K_i}$. We must have $k \leqslant i$ by our assumption about k. Either $Q_{K'_k}Q_{K_i}$ is zero or it is $Q_{K_i+K'_k}$, which is the same form as K_i. Either this is not $(1)_i$ and we have odd powers in our symmetric function and we can ignore this, or it is $(1)_i$ and the lead term we get is distinct from that for $Q_{(1)_k}s_{I_kJ_q}$ above and they cannot cancel out. We have now shown linear independence.

We must show that we get all of $H^*\mathrm{BO}_n$ this way. We simply give a one to one correspondence between symmetric functions and elements $Q_{K_k}s_{I_kJ_q}$, $s_{I_kJ_q} \in G_k$. To go from $Q_{K_k}s_{I_kJ_q}$ to $H^*\mathrm{BO}_n$, we choose the symmetric function on

$$Q_0^{\varepsilon_0}(x_1^{2i_1+1})Q_1^{\varepsilon_1}(x_2^{2i_2+1}) \ldots Q_{k-1}^{\varepsilon_{k-1}}(x_k^{2i_k+1})x_{k+1}^{2j_1} \ldots x_{k+q}^{2j_q}.$$

We need an inverse, that is, to go from a symmetric function to a $Q_{K_k}s_{I_kJ_q}$, $s_{I_kJ_q} \in G_k$. Assume inductively that we have defined $i_1, \ldots, i_v$, $\varepsilon_0, \ldots, \varepsilon_{v-1}$, and $j_1, \ldots, j_m$. We want to define either j_{m+1} or i_{v+1} and ε_v. Recall that we started with some

$$s_T = \sum x_1^{t_1} \ldots x_u^{t_u}.$$

Inductively we have used up $v+m$ of the t. Inductively we assume that all remaining odd t are at least $2i_v+1$ and all the even t must be greater than $2i_v+2^v$.

Case (1). If the next lowest even power, t, is less than $2i_v+2^{v+1}$, we use it to define j_{m+1}.

Case (2). If the next lowest even power, t, is equal to $2i_v+2^{v+1}$ and there are at least two such t, we use two of them to define j_{m+1} and j_{m+2}.

Case (3). If the next lowest even power, t, is equal to $2i_v+2^{v+1}$ and there is only one such t left, we define $i_{v+1} = i_v$ and $\varepsilon_v = 1$.

Case (4). If there are at least two t equal to the next lowest even t, we use them to define j_{m+1} and j_{m+2}.

Case (5). Let t' be the next lowest even power and let t'' be the next lowest odd power. Recall that $t' > 2i_v+2^{v+1}$ and $t'' \geqslant 2i_v+1$. Compare $t'-2^{v+1}+1$ and t''. Use the smaller one for $2i_{v+1}+1$, that is, to define i_{v+1}. If t' is used we set $\varepsilon_v = 1$, and if t'' is used we set $\varepsilon_v = 0$. If $t'-2^{v+1}+1 = t''$ we choose t' to define i_{v+1}.

We leave it to the reader to begin the above induction and to check that these maps are inverses to each other. There is probably some filtration in which this is the E module structure of the associated graded object, but we have not pursued this. As previously discussed, this concludes the proof of part (b) of our lemma.

We have not yet proven that the G_k are E module generators. This fact will come out later during our computation of Ext.

If M is a graded module, we let M^{-*} be the module with negative grading and M^* be the dual module.

THEOREM 2.2. (i) *The Adams spectral sequence*

$$E_2 \simeq \mathrm{Ext}_E^{**}(H^*\mathrm{BO}_n, \mathbb{Z}/2) \Rightarrow \mathrm{BP}_*\mathrm{BO}_n \otimes \mathbb{Z}_2$$

collapses, has

$$\left(\bigoplus_{k=0}^{n} G_k\right)^* \simeq \mathrm{Ext}_E^0(H^*\mathrm{BO}_n, \mathbb{Z}/2)$$

and there is a filtration on Ext *such that the associated graded*

$$\mathrm{BP}'_* \simeq \mathbb{Z}/2[v_0, v_1, \ldots] \simeq \mathrm{Ext}_E(\mathbb{Z}/2, \mathbb{Z}/2)$$

module is

$$\bigoplus_{k=0}^{n} \mathrm{BP}'_*/I'_k \otimes G_k^* .$$

(ii) *The Adams spectral sequence*

$$E_2 \simeq \mathrm{Ext}_E^{**}(\mathbb{Z}/2, H^*\mathrm{BO}_n) \Rightarrow \mathrm{BP}^{-*}\mathrm{BO}_n \otimes \mathbb{Z}_2$$

collapses, has

$$\mathrm{Ext}_E^{0,*}(\mathbb{Z}/2, H^*\mathrm{BO}_n) \simeq \left(\bigoplus_{k=0}^{n} Q_{(1)_k} G_k\right)^{-*} \simeq (H^*\mathrm{BU}_n)^{-*} \subset (H^*\mathrm{BO}_n)^{-*} ,$$

and there is a filtration on Ext *such that the associated graded* BP'_* *module is*

$$\bigoplus_{k=0}^{n} \mathrm{BP}'_*/I'_k \otimes (Q_{(1)_k} G_k)^{-*} .$$

Proof. We define an E-module filtration on $H^*\mathrm{BO}_n$ by $F^sH^*\mathrm{BO}_n$, a sub E-module of $H^*\mathrm{BO}_n$ generated by all $(G_k)^i$, $i \geqslant s-(2^{k+1}-2-k)$. Note that the degree of $Q_{(1)_k}$ is $2^{k+1}-2-k$, so that

$$\bigoplus_{k=0}^{n} Q_{(1)_k}(G_k)^{s-(2^{k+1}-2-k)} = H^s\mathrm{BU}_n \subset H^s\mathrm{BO}_n .$$

Also, the degree of Q_k is $2^{k+1}-1 > 2^{k+1}-2-k$. We see that, as E-modules, considering E_k as a quotient module,

$$F^s/F^{s+1} \simeq \bigoplus_{k=0}^{n} E_k(G_k)^{s-(2^{k+1}-2-k)} .$$

This does not depend on the G_k being generators, a fact we have not yet proven. The computation of both Ext groups for this associated graded module can now be carried out using 1.11. We get the answers described in Theorem 2.2. We must show that several spectral sequences collapse. For $\mathrm{BP}^*\mathrm{BO}_n$, the spectral sequence for the filtration 1.13, and the Adams spectral sequence are both in even degrees and therefore collapse. We use Ravenel's trick to deduce the collapse for the corresponding spectral sequences for $\mathrm{BP}_*\mathrm{BO}_n$. If the spectral sequence of the filtration and the Adams spectral sequence for $\mathrm{BP}_*\mathrm{BO}_n$ both collapse, we can use the filtration we obtain for $\mathrm{BP}_*\mathrm{BO}_n$ to compute the Ext for the duality spectral sequence 1.14:

$$\mathrm{Ext}_{\mathrm{BP}_*}(\mathrm{BP}_*\mathrm{BO}_n, \mathrm{BP}^*) \Rightarrow \mathrm{BP}^*\mathrm{BO}_n .$$

From 1.15, we see that the total degree

$$\mathrm{Ext}_{\mathrm{BP}_*}\left(\bigoplus_{k=0}^{n} \mathrm{BP}_*/I_k \otimes G_k^*, \mathrm{BP}^*\right) \simeq \bigoplus_{k=0}^{n} \mathrm{BP}^*/I_k \otimes (Q_{(1)_k} G_k)$$

(where we have abused notation for $k = 0$). This is the desired result. The only way we can obtain this is if both spectral sequences of filtration 1.12, the duality spectral sequence, and the $\mathrm{BP}_*\mathrm{BO}_n$ Adams spectral sequence all collapse. This concludes our proof of Theorem 2.2, and it implies that the G_k are E-module generators and concludes the proof of Theorem 2.1.

3. *Elegance dismantled*

As we have seen from the Introduction and Section 2, the algebra

$$\mathrm{BP}^*[[C_1, \ldots, C_n]]/(C_1 - C_1^*, \ldots, C_n - C_n^*)$$

maps surjectively to $\mathrm{BP}^*\mathrm{BO}_n$. We shall dismantle this algebra and show that it must inject. In Section 2 we showed that all symmetric functions of even powers could be written as

$$\sum x_1^{2i_1+2} \ldots x_k^{2i_k+2^k} x_{k+1}^{2j_1} \ldots x_{k+q}^{2j_q}, \qquad k+q \leqslant n$$

with $0 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k$ and $0 < j_1 \leqslant \ldots \leqslant j_q$, where if the number of j equal to j_u is odd, then there is some s such that

$$2i_s + 2^s < 2j_u < 2i_s + 2^{s+1} .$$

In the filtration of Theorem 2.2(ii), this generator is associated with BP^*/I_k. Viewed as a symmetric function in $\mathrm{BP}^*\mathrm{BU}_n \subset \mathrm{BP}^* \times {}^n\mathbb{C}P^\infty$, we replace x_i^2 with T_i. The proof of Theorem 1 will be concluded with the following.

THEOREM 3.1. *Let $p = 2$. There is a* BP* *module filtration of the algebra*

$$\mathrm{BP}^*[[C_1, \ldots, C_n]]/(C_1 - C_1^*, \ldots, C_n - C_n^*)$$

such that the generators of the associated graded object are given by a basis for the symmetric functions $\mathrm{BP}^*\mathrm{BU}_n \subset \mathrm{BP}^* \times {}^n\mathbb{C}P^\infty$ *and the ideal* I_k *kills the generator*

$$\sum T^{I_k J_q} = \sum T_1^{i_1+1} \dots T_k^{i_k+2^{k-1}} T_{k+1}^{j_1} \dots T_{k+q}^{j_q}, \qquad k+q \leqslant n$$

with $0 \leqslant i_1 \leqslant i_2 \dots \leqslant i_k$ *and* $0 < j_1 \leqslant \dots \leqslant j_q$, *where if the number of* j *equal to* j_u *is odd, then there is some* s *such that*

$$i_s + 2^{s-1} < j_u < i_s + 2^s .$$

REMARK. We regret that the notation I and I_k is used with more than one meaning. They should be clear from context.

Proof. We have an exact sequence

$$0 \longrightarrow (C_n) \longrightarrow \mathrm{BP}^*[[C_1, \dots, C_n]]/(C_k - C_k^*)$$
$$\longrightarrow \mathrm{BP}^*[[C_1, \dots, C_{n-1}]]/(C_k - C_k^*) \longrightarrow 0 .$$

By induction on n, we need only prove the result for $k+q = n$, which is (C_n). Let T^K be the lead term of the $(k+q = n)$-symmetric function in 3.1. Let

$$L = (i_1+1, i_2+1, \dots, i_k+1, j_1, \dots, j_{2v}, j_{2v+1} - j'_1, \dots, j_{2v+q'} - j'_{q'}),$$

where $j_1, \dots, j_{2v}$ are the even number of j less than or equal to i_1+1. We have $2v+q' = q$. If $i_s + 2^{s-1} \leqslant j_u < i_{s+1} + 2^s$, let $j'_u = j_u - (i_s+1)$, $s < k$. If $j_u \geqslant i_k + 2^{k-1}$, then $j'_u = j_u - (i_k+1)$. From 1.23 we see that

3.2 $$s_L \sum T_1^{2-1} T_2^{2^2-1} \dots T_{k-1}^{2^{k-1}-1} T^{j'_1} \dots T^{j'_{q'}} = s_K$$

plus symmetric functions with higher lead terms

where if there are an odd number of j' equal to j'_u, then there is $s \leqslant k$ such that $2^{s-1}-1 < j'_u < 2^s - 1$. Because $j_1 > 0$ and $j_u - j'_u > 0$, $u > 2v$, we have that C_n divides T^L and thus s_L. Observe also that $n - (q'+k-1)$ is always odd; $n = k+q$, $q = 2v+q'$, and so $n-(q'+k-1) = 2v+1$. Let

$$I = (i_1, \dots, i_r), \qquad i_k \neq 0,\ r < n .$$

$$I_j = (i_1, \dots, \hat{i}_j, \dots, i_r)$$

where we delete i_j, and

$$\Delta_k = (0, \dots, 1, 0, \dots),$$

with a figure one in the k-th coordinate. We have

$$s_I = \sum T_1^{i_1} \dots T_r^{i_r} .$$

We compute, from 1.30 (with $I = (2, v_1, \dots)$) the right-hand side of

$$0 = e_0 s_I + \sum_{k=1}^{r} e_{i_k} s_{I_k} \bmod \left(I^2, (v_{n+1}, v_{n+2}, \dots)\right).$$

We write down more terms than are really there for simplification of notation, but the computation works out the same. We have

$$3.3 \qquad 0 = \left(1-(-1)^n + \sum_{n \geqslant i > 0} v_i s_{2^i-1}\right) s_I + \sum_{k=1}^{r} \sum_{n \geqslant i \geqslant 0} v_i s_{2^i-1+i_k} s_{I_k}$$

$$= (1-(-1)^n) s_I + \sum_{n \geqslant i > 0} v_i \sum_{k=1}^{r+1} s_{I+(2^i-1)\Delta_k} + \sum_{k=1}^{r} \sum_{n \geqslant i \geqslant 0} v_i \sum_{q=1}^{r} s_{I_k+(2^i-1+i_k)\Delta_q}$$

$$= (1-(-1)^n) s_I + \sum_{n \geqslant i > 0} v_i s_{I+(2^i-1)\Delta_{r+1}} + v_0(r) s_I ,$$

where the sum is over all i such that the number of i_k equal to 2^i-1 is even. Let $I = (1, 3, \ldots, 2^{k-1}-1, j'_1, \ldots, j'_{q'})$ as above, where if $j'_u = 2^i-1, i < k$, then there are an even number of such j'_u. Because $n-(q'+k-1)$ is always odd, this gives

$$3.4 \qquad 2s_I = 0 \bmod (I^2, (v_1, v_2, \ldots)) .$$

Let $i < k$ and

$$I_i = (1, 3, \ldots, \widehat{2^i-1}, \ldots, 2^{k-1}-1, j'_1, \ldots, j'_{q'}) ;$$

we get

$$3.5 \qquad v_i s_I = 0 \bmod (I^2, (v_{i+1}, v_{i+2}, \ldots)) .$$

Multiply by s_L from 3.2 to get, for $i < k$,

$$3.6 \qquad v_i \sum T^{I_k J_q} \in I^2 + (v_{i+1}, v_{i+2}, \ldots)$$

$$+ v_i \text{ symmetric functions with higher lead terms.}$$

This is the formula we use to show 3.1. We just have to filter our algebra right. Let G_k be generated by the symmetric functions of 3.1 (with $k+q = n$). Then $\bigoplus_{k=0}^{n} G_k$ generates all of (C_n). There is nothing to prove for G_0, so we can mod it out now.

Let F^s be the BP* submodule of $(C_n)/\mathrm{BP}^* G_0$ generated by

$$(G_i)^t, \quad t \geqslant s+2(2^i-1), \, i > 0 .$$

We want to show that

$$I_i G_i^{s+2(2^i-1)} = 0 \bmod F^{s+1} .$$

This will complete our proof. Fix an arbitrary s'. It is enough to prove the formula in $(C_n)/(\mathrm{BP}^* G_0 + F^{s'})$ using downward induction on s. In here, of course, there is nothing to prove for $s \geqslant s'$. For all $s < s'$, $F^{s+1} \supset F^{s'}$, and so the use of s' becomes irrelevant; it only allows us to begin our induction. What we prove is that

$$3.7 \qquad I_i (G_i \oplus G_{i+1} \oplus \ldots \oplus G_n)^{s+2(2^i-1)} = 0 \bmod F^{s+1} .$$

By downward induction on s, 3.7 gives us

$$I_{i-1} (G_i \oplus G_{i+1} \oplus \ldots \oplus G_n)^{s+2(2^i-1)} = 0 \bmod F^{s+1} ,$$

and so we want to show that

3.8 $$v_{i-1}(G_i \oplus \ldots \oplus G_n)^{s+2(2^i-1)} = 0 \bmod F^{s+1}.$$

From 3.6 we have, for $i-1 < k$, $\sum T^{I_k J_q} \in G_k$,

3.9 $$v_{i-1} \sum T^{I_k J_q} \in I^2 + (v_i, v_{i+1}, \ldots) + v_{i-1} \text{ symmetric functions with higher lead terms.}$$

The degree of 3.8 is $s+2(2^i-1)-2(2^{i-1}-1) = s+2^i$, and we are only concerned with $G_i \oplus \ldots \oplus G_n$ in this degree because for $k < i$, $s+2^i > s+2(2^k-1)$. We take care of the terms of 3.9. We can prove the result by downward induction on the lead terms and so we can ignore the v_{i-1} symmetric functions with higher lead terms. Consideration of the $(v_i, v_{i+1}, \ldots)$-terms shows that we may have a term in $(k \geqslant i > 0)$:

$$v_k(G_i \oplus \ldots \oplus G_n)^{s+2^i+2(2^k-1)}.$$

But $G_j^{s+2^i+2(2^k-1)} \subset F^{s+1}$ for $i \leqslant j \leqslant k$ and $v_k G_j^{s+2^i+2(2^k-1)} = 0$, $j > k$ by downward induction on s. Our final concern is with I^2. By the preceding argument we can reduce this to $(I_i)^2$. We already have that I_{i-1} gives zero by induction. So we must have a v_{i-1}^2 term here. However, we have by induction in this degree that v_{i-1} is zero. The only possible problem is that this shows that

$$2(G_1 \oplus \ldots \oplus G_n)^{s+2}$$

must be divisible by $4 \in I^2$. This is a torsion group and so $2(G_1 \oplus \ldots \oplus G_n)^{s+2} = 0$.

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