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Introduction

This paper represents the author's failed attempt to prove the conjectures of Bendersky and Davis, **[B-D]**. They apply unstable BP operations to the study of nondesuspensions of truncated real projective space. As immediate corollaries they obtain nonimmersion results for real projective spaces. Unfortunately, most of their work remains conjectural. Let $\alpha(m)$ be the number of one's in the binary expansion of m. Strong motivation to study **[B-D]** is supplied by their elegant conjecture:

Conjecture 1(s) ([B-D]). *If* $\alpha(m)=s+1$, *then*

$$RP^{2(m+s)} \not\subseteq \boldsymbol{R}^{4m-2s-2}\,. \qquad \square$$

Bendersky and Davis initially proved this conjecture and all of the remaining ones for $s\leqslant 5$. Using our approach and computer tables supplied by Don Davis, we extended this to $s\leqslant 7$. However, Bendersky and Davis really did compute $s=6$, they were just overlooking a term in their preprint. Our proof of $s=7$ does not use further calculations, but is actually a theorem which depends on the $s=6$ case being true in a strong way. The cases, $s=8$ (by computation) and $s=9$ (by theory from $s=8$), are within reach but there is no motivation to carry out these computations. Don Davis informs us that the lowest dimensional new nonimmersion we get is for $m=493$, $\alpha(m)=7$, which gives $RP^{998}\not\subseteq \boldsymbol{R}^{1958}$, an improvement of two over **[A-D]**.

Let $\mathrm{gd}(E)$ be the geometric dimension of the bundle E; $\nu_2(n)$, the number

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of powers of 2 that divides n; ξ_k, the canonical line bundle over RP^k. Conjecture 1(s) is implied by Conjecture 2(s).

Conjecture 2(s) ([B-D]). *If* $d>6s$ *and* $\nu_2\binom{a+s}{d-s}=s$, *then* $\mathrm{gd}((2a+1)\xi_{2d})>2d-6s-2$. □

They show that this, in turn, is a direct consequence of their Conjecture 3(s). Let RP^n_k be the cofiber of $RP^{k-1}\hookrightarrow RP^n$.

Conjecture 3(s) ([B—D]). *If* $d>6s$ *and* $\nu_2\binom{a+s}{d-s}=s$, *then*

$$\nexists\ \Sigma^{2(d-a)-6s-3}RP^{2(a+d)+1}_{2a+1} \qquad \square$$

In a difficult reduction, Bendersky and Davis deduce Conjecture 3(s) from a purely algebraic conjecture involving unstable Brown-Peterson homology information. We need some terminology. The BP cooperations are

$$(4) \qquad BP_*BP\simeq BP_*[h_1, h_2, \cdots], \text{ where } |h_n|=2(p^n-1)\,,$$

and $h_n=c(t_n)$, see [A]. For $J=(j_1, j_2, \cdots)$ we write $h^J=h_1^{j_1}h_2^{j_2}\cdots$ and define the length of J, $l(J)=\sum j_i$. Let $[2](x)=\sum a_i x^{i+1}$ be the standard two sequence. The $p=2$ reduced Brown-Peterson homology of RP^∞_{2b+1} is generated as a BP_* module by $\gamma_k\in BP_{2k+1}RP^\infty_{2b+1}$, $k\geqslant b$. The only relations are

$$(5) \qquad \sum_{i=0}^{k-b} a_i\gamma_{k-i}=0 \qquad (a_0=2,\ a_1=u_1)\,.$$

Bendersky and Davis define $V_b\subset BP_*BP\otimes_{BP_*}BP_*RP^\infty_{2b+1}$ as the left BP_* module generated by all $h^J\otimes u\gamma_c$ with $2l(J)\leqslant|u\gamma_c|$. Then they define

$$(6) \qquad W_b\equiv BP_*BP\otimes_{BP_*}BP_*RP^\infty_{2b+1}/V_b\,.$$

Note that W_b is a left-right BP_* module. Bendersky and Davis show that Conjecture 7(s) implies Conjecture 3(s).

Conjecture 7(s) ([B-D]). *For b large* $(2b\geqslant 3s+1)$,

$$0\neq 2^s h_1^{b+2s+1}\otimes\gamma_{b+s}\in W_b\,. \qquad \square$$

It is fairly easy to see that 7(s) implies 7($s-1$). We show this later. The module W_b is where Bendersky and Davis do their calculations. In fact, they set h_2, h_3, etc. equal to zero in W_b and can still show 7(s) for $s\leqslant 6$.

Theorem 8 (Bendersky-Davis, [B-D]). *Conjecture* 7(s) *is true for* $s\leqslant 6$ *and Conjecture* 7(s)$\Rightarrow$*Conjecture* 3(s)$\Rightarrow$*Conjecture* 2(s)$\Rightarrow$*Conjecture* 1(s). □

Rather than to show directly that the elements of W_b are nonzero, we have concentrated on describing the entire structure of W_b through the necessary range. A few calculations make this structure clear. Its simplicity pulled the author into believing he would be able to prove the conjectures of **[B-D]**. Difficulties arose. The attempt to prove it almost works, but the way is blocked by one troublesome element. We describe this structure in Section 2. This leads to our version of the conjecture. We have

$$(9) \qquad BP_* \simeq Z_{(p)}[u_1, u_2, \cdots], \quad |u_n| = 2(p^n-1).$$

As usual, **[J-W]**, let $BP\langle 2\rangle_* \equiv BP_*/(u_3, u_4, \cdots)$. Define

$$(10) \qquad B' \equiv BP\langle 2\rangle_* \otimes_{BP_*} BP_*BP/(h_2, h_3, \cdots).$$

Let x have degree -2 and define, for $p=2$,

$$(11) \qquad B \equiv h_1^{-1} B' \otimes_{BP_*} \widetilde{BP_*[[x]]/([2](x))}.$$

Define $R \subset B$ as the left $BP\langle 2\rangle_*$ module generated by all negative degree elements $a \otimes cx^i$ with $|a| \leqslant |cx^i|$ except those of degree -2 with $i=1$.

Conjecture 12. $0 = 2(1 \otimes x) \in B/R$. □

Our main result is

Theorem 13. (a) *Conjecture* 12 *implies Conjecture* 7 *for all s.*

(b) *If* $0 = 2(1 \otimes x) \in B/R$ *modulo*(x^{3k+2}), *then Conjecture* 7(*s*) *is true for* $s \leqslant 2k+1$.

(c) $0 = 2(1 \otimes x) \in B/R$ *modulo*(x^{13}). □

Part (c) implies all of the previous conjectures for $s \leqslant 7$. It was proven by computations not represented in this paper.

In the process of our investigation, the idea of BP metastability became clear and we hope it will be of independent interest. Let $\{\underline{BP}_n\}$ be the Ω-spectrum for BP, see **[R-W]** or **[W]**. Using the homology suspension, we can identify $QBP_*\underline{BP}_n$ with its image in BP_*BP. We get

$$(14) \quad QBP_*\underline{BP}_{2n-1} \subset QBP_*\underline{BP}_{2n} \simeq QBP_*\underline{BP}_{2n+1} \subset QBP_*\underline{BP}_{2n+2} \subset \cdots BP_*BP.$$

Theorem 15 (the metastable range). *Let* $M_n \equiv BP_*BP/QBP_*\underline{BP}_{2n}$, $n \geqslant 0$.

(a) *Multiplication by* h_1 *defines a left-right* BP_* *module map*

$$\Sigma^{2(p-1)} M_{n-1} \rightarrow M_n,$$

(b) *which is injective for degree* $s \leqslant 2(p^2-1)n$,

(c) *and surjective for degree* $s < 2(p^2-1)(n+1)$. □

BP_*BP, and thus M_n, are "sparse", i.e., they are nontrivial only in degrees which are multiples of $2(p-1)$. If we use "hyper-complex" grading (divide the real grading by $2(p-1)$) we can rephrase Theorem 15 as $\Sigma M_{n-1} \to M_n$ is an isomorphism in "hyper-complex" degrees $\leqslant (p+1)n$, which looks more like metastability. The next result is a corollary of our study of the M_n.

Corollary 16 (the metastable BP_* module). *Let* $M \equiv \varinjlim \Sigma^{-2(p-1)n} M_n$.

(a) *M is generated as a left BP_* module by the h^J, $l(J)>0$, where $j_1 \in Z$.*

(b) *The element h^J has order $p^{l(J)}$.*

(c) *There is a filtration of M with associated graded object E_0M, with $(p)E_0M=0$.*

(d) *E_0M is free as a $BP_*/(p)$ module on the h^J, $l(J)>0$.* □

Define

$$M_n(k) = M_n/(h_1 M_{n-1} + \cdots + h_{k-1} M_{n-1}) .$$

We have a generalized metastable range.

Theorem 17. *Multiplication by* h_k,

$$h_k \colon \Sigma^{2(p^k-1)} M_{n-1}(k) \to M_n(k) ,$$

induces a left-right BP_* module map which is

(a) *injective for degree* $s < 2(p^{k+1}-1)n + 2(p^k-1)$, *and*

(b) *surjective for degree* $s < 2(p^{k+1}-1)(n+1)$. □

An analog of Corollary 16 is easy to state, and if true, is more difficult to prove, and, at present, of far less interest than the metastable module.

We would like to thank Martin Bendersky and Don Davis for sharing their insights into this problem. This paper owes a large debt to their work. Also, our computation of the $s=6$ and 7 cases would have been impossible without the tables supplied by Don Davis. We thank Douglas Ravenel and Hirotaka Tamanoi for many conversations about the problem. Thanks also to Professor Shimada and the RIMS, Kyoto University, for their hospitality during much of this work. We gratefully acknowledge the generous support of the Johns Hopkins University, the N.S.F., and Ministry of Education in Japan.

We prove our metastable results in Section 1 and we fill Section 2 with our thoughts on the Bendersky-Davis Conjectures.

§ 1. The Metastable Range

When we need specific generators for BP_*, we use Araki's [**Ar**]. Their main property is:

$$[p](x) = \sum_{i\geq 0}^{F} u_i x^{p^i} = \sum_{i\geq 0} a_i x^{i+1}, \qquad (a_0 = p = u_0,\ a_{p-1} = u_1)\,. \tag{1.1}$$

To compute the right unit we have

$$\sum_{i,j\geq 0}^{R} u_i^{p^j} h_j x^{p^{i+j}} = \sum_{i,j\geq 0}^{R} h_i^{p^j} u_j x^{p^{i+j}}\,, \tag{1.2}$$

where the R indicates the formal group sum with coefficients written on the right. This follows as in [**R**], see [**B-D**]. We use a similar notation, L, for the formal group law written on the left. Sometimes we use F for L or when there is no difference between left and right. There is a different, but equally frustrating, formula for the right unit.

Proposition 1.3. $\sum_{i\geq 0}^{R} (\sum_{j\geq 0}^{L} u_j x^{p^j})^{p^i} h_i = \sum_{i\geq 0}^{R} (\sum_{j\geq 0}^{R} h_j x^{p^j})^{p^i} u_i.$ □

Proof. We use the formula, Theorem 11.111, p. 80, of [**W**]. After re-arranging and inserting an x, it reads

$$\sum_{i\geq 0} (a(x))^{i+1} b_i = \sum_{i\geq 0} (b(x))^{i+1} a_i \quad \text{where}$$
$$a(x) = \sum_{i\geq 0} a_i x^{i+1} = \sum_{i\geq 0}^{F} u_i x^{p^i}, \quad \text{and}$$
$$b(x) = \sum_{i\geq 0} b_i x^{i+1} = c(\sum_{i\geq 0}^{F} t_i x^{p^i}) = \sum_{i\geq 0}^{R} h_i x^{p^i}\,.$$

The result follows by substitution.

Remark. There is a misprint in 11.111. The F was left out of $c(\sum^F t_i)$.

Bendersky and Davis observe that [**R-W**] immediately implies:

Corollary 1.4. *$QBP_*\underline{BP}_{2n} \subset BP_*BP$ is the left BP_* module generated by all $h^J u$, $2l(J) - |u| \leqslant 2n$, $u \in BP_*$.* □

We see that $QBP_*\underline{BP}_{2n} \subset BP_*BP$ is also a right BP_* submodule by this. It is not really necessary to consider all of the above $h^J u$ because $QBP_*\underline{BP}_{2n}$

has been thoroughly studied. It is a free left BP_* module and two different bases have been found. For this work we prefer Boardman's basis, ([**B**]), over the original, ([**R-W**]). We stabilize Boardman's basis. Let $u^I = u_1^{i_1} u_2^{i_2} \cdots$.

Theorem 1.5 (Boardman's basis, [B]). *The left BP_* module $QBP_*\underline{BP}_{2n} \subset BP_*BP$ is free on generators:*

$$h^J u^I ,$$

such that if

$$I = \Delta_{k_0} + \Delta_{k_1} + \cdots + \Delta_{k_m} + I' , \qquad k_0 \leqslant k_1 \leqslant \cdots ,$$

I' a nonnegative sequence, then

$$j_m < p^{k_m} , \quad m > 0 , \quad and$$
$$2n - 2p^{k_0} < 2l(J) - |u^I| \leqslant 2n . \qquad \square$$

Proof. Boardman's basis is for the spaces, not after stabilization. We have elements $\bar{h}_i \in BP_{2p^i}\underline{BP}_2$, $i \geqslant 0$. For $i > 0$, $\bar{h}_i$ stabilizes to h_i while $\bar{h}_0$ stabilizes to $1 \in BP_*BP$. Boardman's basis for $QBP_*\underline{BP}_{2n}$ is, where $J = (j_0, j_1, \cdots)$, $\bar{h}^J u^I$, such that if I is as above, then $j_m < p^{k_m}$, $m \geqslant 0$, and $l(J) - |u^I| = 2n$. Since $\bar{h}_0^{j_0}$ stabilizes to 1, we must alter the condition on j_0 to the new condition in Theorem 1.5. $\square$

Proof of Theorem 15 (*a*). Multiplication by h_1 is a left-right BP_* module map of BP_*BP to itself which raises degree by $2(p-1)$. Using 1.4 we see that h_1 induces a map of $QBP_*\underline{BP}_{2(n-1)}$ to $QBP_*\underline{BP}_{2n}$, which are left-right BP_* modules. $\square$

Proof of Theorem 15 (*b*). Define $M'_n \equiv BP_*BP/BP_*\{h^J \mid l(J) \leqslant n\}$. We have the commuting diagram

$$\begin{array}{ccccc}
0 \longrightarrow & \Sigma^{2(p-1)} M'_{n-1} & \xrightarrow{h_1} & M'_n \\
 & \downarrow & & \downarrow \\
 & \Sigma^{2(p-1)} M_{n-1} & \xrightarrow{h_1} & M_n \\
 & \downarrow & & \downarrow \\
 & 0 & & 0 .
\end{array}$$

The lowest degree relation, $h^J u^I$, $I \neq 0$, defining M_n, which is not divisible by h_1 is, by 1.5, $h_2^n u_1$. This is in degree $2(p^2-1)n + 2(p-1)$. We show we have an injection for lower degree. Let $x \in M'_{n-1}$ such that $h_1 x$ reduces to zero in

M_n and is of degree such that all $h^J u^I$, $I \neq 0$, are divisible by h_1. Then $h_1 x = \sum a_{IJ} h^J u^I$, and by injectivity $x = \sum a_{IJ} h^{J-\Delta_1} u^I$ and x reduces to zero in M_{n-1}.

Proof of Theorem 15(*c*). M'_n is generated by h^J, $l(J) > n$, so M_n is also. The lowest degree h^J with $j_1 = 0$ and $l(J) > n$ is h_2^{n+1} in degree $2(p^2-1)(n+1)$. The result follows. □

Proof of Theorem 17. Define $M'_n(k)$ to be $M'_n / BP_* \{h^J \mid j_1 + \cdots + j_{k-1} > 0\}$. Then the proof is the same as for 15 except that we must observe that $h_{k+1}^n u_i$ is already zero in $M'_n(k)$ for $i < k$. □

We can now begin a more detailed study of the metastable range.

Lemma 1.6. *In degree* $s \leqslant 2(p^2-1)n$, *the left* BP_* *module generators of* $QBP_* \underline{BP}_{2n} \subset BP_* BP$ *are given by all*

$$h^J, \ l(J) \leqslant n \quad \textit{and all}$$
$$h^J u_i, \quad i > 0, \quad 0 \leqslant l(J) - n < p^i . \qquad \square$$

Proof. If we take a Boardman basis element from 1.5 with two or more u's, say $h^J u_{k_0} u_{k_1} u^{I'}$, then $j_1 < p^{k_1}$ and our lowest degree h^J with this a possible Boardman basis element has $h^J = h_1^{p^{k_1}-1} h_2^{j_2}$, with $2n - 2p^{k_0} < 2l(J) - |u^I| \leqslant 2l(J) - |u_{k_0} u_{k_1}| = 2(p^{k_1}-1) + 2j_2 - 2(p^{k_0}-1) - 2(p^{k_1}-1) = 2j_2 - 2(p^{k_0}-1)$, so $2n-2 < 2j_2$, i.e., $j_2 = n$, but this is out of our range. So we can only have one u. In this case the first condition in 1.5 is empty. The second condition becomes $2n - 2p^i < 2l(J) - |u_i| \leqslant 2n$, which easily leads to our condition. □

Definition 1.7. Order the J by, $J' < J$ if $l(J') < l(J)$. If $l(J') = l(J)$, then $J' < J$ if $j'_k > j_k$ when $j_{k+i} = j'_{k+i}$, $i > 0$. We define a filtration of $BP_* BP$ indexed by the J. Let $F_J BP_* BP$ be the left BP_* module generated by all $h^{J'}$, $J' < J$. This induces a filtration, $F_J M_n$, of M_n. Let $E_0 M_n$ denote the associated graded object. Since we will always be working in a finite range our filtration is finite.

We need a fact about the right unit.

Lemma 1.8. *For* $k > 0$, $J = ((p^k - p)/(p-1), 0, 0, \cdots)$, $\eta(u_k) = (-1)^k (p h_1 - u_1) h_1^{(p^k-p)/(p-1)}$ *modulo* $(p^2, pu_1, F_J BP_* BP)$. □

Proof. Let $I = (p, u_1, u_2, \cdots)$. In this degree we are working modulo $(I^2, h_2, h_3, \cdots)$. Modulo I^2, the formal group sums of 1.2 become ordinary sums. Modulo $(h_2, h_3, \cdots)$ the coefficient of x^{p^k} is

$$u_k = \eta(u_k) + h_1^{p^{k-1}} u_{k-1} .$$

We prove the result by induction. For $k=1$ we have $\eta(u_1)=u_1-ph_1$. For $k>1$, u_k on the left implies we are in $F_J BP_*BP$ for dimensional reasons, so our formula degenerates to $\eta(u_k)=-h_1^{p^{k-1}}u_{k-1}$. The result follows by induction. □

Proposition 1.9. *In degree* $s\leqslant 2(p^2-1)n$,

(a) $pE_0M_n=0$.
(b) E_0M_n *is* $BP_*/(p)$ *free on the* h^J, $l(J)>n$. □

Proof. From 1.8, the defining relations for M_n, as given in 1.6, are represented in E_0BP_*BP by

$$h^J,\ l(J)\leqslant n\,, \quad \text{and} \quad (-1)^i ph^{J+k_i\Delta_1} \bmod (p^2), \quad i>0\,,$$
$$0\leqslant l(J)-n<p^i\,,$$

where $k_i=(p^i-p)/(p-1)+1=(p^i-1)/(p-1)$.

For part (a) it is enough to show that the $h^{J+k_i\Delta_1}$ include all $h^{J'}$, $l(J')>n$. We know that $k_i+p^i=k_{i+1}$, $k_1=1$. Find i such that $k_i\leqslant l(J')-n<k_{i+1}$. Then $0\leqslant l(J')-n-k_i<p^i$. We claim that $j_1'\geqslant k_i'$. If not, the lowest degree $h^{J'}$ satisfying our conditions is $h_1^{k_i-1}h_2^{n+1}$ which is out of our metastable range. We can let $J=J'-k_i\Delta_1$. This concludes part (a).

For part (b) we must show that the $J+k_i\Delta_1$ are all distinct and $l(J+k_i\Delta_1)>n$. We are given that $l(J)\geqslant n$ and $k_i>0$, so $l(J+k_i\Delta_1)>n$. The argument in part (a) shows that the $J+k_i\Delta_1$ are all distinct. □

Proposition 1.10. *In degree* $s\leqslant 2(p^2-1)n$ *in* M_n, *the order of the generators* h^J, $l(J)>n$, *is* $p^{l(J)-n}$. □

Proof. We show that

$$p^{l(J)-n}h^J=0 \quad \text{and}$$
$$p^k h^J = u_1^k h^{J-k\Delta_1} \quad \text{modulo} \quad F_{J-k\Delta_1}M_n\,, \qquad k=l(J)-n-1\,.$$

This last element is nonzero by 1.9(b) so these prove the result. From 1.6 and 1.8 we can deduce, for arbitrary J, that

$$ph^J = u_1h^{J-\Delta_1} \quad \text{modulo} \quad (p^2h^J,\, pu_1h^{J-\Delta_1},\, F_{J-\Delta_1}M_n)\,.$$

By induction on the order of J, $p^kF_{J-\Delta_1}M_n=0$ and $p^kh^{J-\Delta_1}=0$, so we are left with $p^{l(J)-n}h^J=ap^{l(J)-n+1}h^J$, so $p^{l(J)-n}h^J=0$.

By induction on J, $p^{k-1}F_{J-\Delta_1}M_n\subset F_{J-k\Delta_1}M_n$, $p^ku_1h^{J-\Delta_1}=0$, and $p^{k+1}h^J=0$, so

$p^k h^J = p^{k-1}(ph^J) = p^{k-1}(u_1 h^{J-\Delta_1}) = u_1(p^{k-1}h^{J-\Delta_1}) = u_1^k h^{J-k\Delta_1}$ modulo $F_{J-k\Delta_1}M_n$ by induction. □

A convenient notation for $\Sigma^{-2(p-1)n}M_n$ in Corollary 16 is to subtract n from j_1 in M_n. Corollary 16 now follows from 1.9 and 1.10.

§ 2. The Structure of W_b

We want to take advantage of the Boardman basis in our analysis of W_b. In this section, $p=2$.

Lemma 2.1. $V_b = \sum_{c\geqslant b} QBP_*\underline{BP}_{2c+1}\otimes\gamma_c \subset BP_*BP\otimes_{BP_*}BP_*RP^{\infty}_{2b+1}$. □

Proof. $QBP_*\underline{BP}_{2c+1} \simeq QBP_*\underline{BP}_{2c}$. From Corollary 1.4, this is generated as a left BP_* module by all $h^J u$, $2l(J)-|u|\leqslant 2c$, or $2c+1$. So the above group is the left BP_* module generated by $h^J u\otimes\gamma_c = h^J\otimes u\gamma_c$, $2l(J)\leqslant|u|+2c+1=|u\gamma_c|$, the same as the definition of V_b before (6) in the introduction. □

Remark. Bendersky and Davis obtain their definition from this fact, their conjecture 7 takes place only in the metastable range of $QBP_*\underline{BP}_{2c}$, so we take a limit similar to the definition of M in Section 1.

The defining relations, (5), show that $\gamma_c\to\gamma_{c+1}$ gives an isomorphism

$$\widetilde{BP}_*RP^{\infty}_{2(b-1)+1} \to \widetilde{BP}_*RP^{\infty}_{2b+1}\,. \tag{2.2}$$

We extend this to a left-right BP_* map, $\Sigma^4 W_{b-1}\to W_b$, given by taking $h^J\otimes\gamma_c$ to $h^{J+\Delta_1}\otimes\gamma_{c+1}$. In our range of interest we can use the results of Section 1. This map takes generators to generators and relations among them to relations, all in a one to one fashion similar to the work in Section 1. We have an isomorphism in our range.

Define

$$W = \lim_{b\to\infty}\Sigma^{-4b-1}W_b\,. \tag{2.3}$$

We alter our notation so

$$\lim_{b\to\infty} h^{J+b\Delta_1}\otimes\gamma_{b+c} \equiv h^J\otimes\gamma_c \tag{2.4}$$

$c\geqslant 0$, $|\gamma_c|=2c$, $j_1\in Z$. The defining relations are now $h^J\otimes u\gamma_c$, $2l(J)\leqslant|u|+2c$. The lowest degree non zero element is $\lim h_1^{(b+1)}\otimes\gamma_b = h_1\otimes\gamma_0$, in degree 2. We can rephrase the Bendersky-Davis Conjecture 7 as.

Conjecture 2.5. $0\neq 2^s h_1^{2s+1}\otimes\gamma_s\in W$. □

Lemma 2.6. *In W,*

$$2^s h_1^{2s+1}\otimes r_s = h_1^{2s+1}\otimes u_1^s r_0 = u_1^s h_1^{2s+1}\otimes r_0 = u_2^s h_1\otimes r_0\,. \qquad \square$$

Remark. This shows that 7(*s*) implies 7(*s*−1).

Proof. We show that $2^{s+1}r_s=0$, by induction on s, using (5).

$$2^{s+1}r_s = 2^s(2r_s) = -2^s(\sum_{i>0} a_i r_{s-i}) = 0\,,$$

by induction. Now, we show $2^s r_s = u_1^s r_0$ by induction on s.

$$2^s r_s = 2^{s-1}(2r_s) = -2^{s-1}(\sum_{i>0} a_i r_{s-i}) = -2^{s-1}a_1 r_{s-1} = -a_1 u_1^{s-1} r_0\,.$$

Since $a_1=u_1$ and $2r_0=0$, we are done. This proves the first equality. Since $\eta(u_1)=u_1$ modulo (2), the second equality follows.

Modulo (2), $\eta(u_2)=u_2+u_1h_1^2+u_1^2h_1$. We show that $u_1^s h_1^k\otimes r_0=0$, $k\leqslant 2s$. We use induction on both s and k. If $k\leqslant 2s-2$ we are done by induction on s, if $k=2s-1$, then

$$(u_1^{s-1}h_1^{2s-3}\otimes r_0)u_2 = u_2(u_1^{s-1}h_1^{2s-3}\otimes r_0)+u_1^s h_1^{2s-1}\otimes r_0+u_1^{s+1}h_1^{2s-2}\otimes r_0\,.$$

All terms are zero by induction except $u_1^s h_1^{2s-1}\otimes r_0$ which must therefore also be zero. The $k=2s$ case follows similarly. We now show the last equality of 2.6, by induction on s.

$$\begin{aligned} u_2^s h_1\otimes r_0 &= u_2(u_2^{s-1}h_1\otimes r_0)\\ &= u_2(u_1^{s-1}h_1^{2s-1}\otimes r_0)\\ &= (u_1^{s-1}h_1^{2s-1}\otimes r_0)u_2+u_1^s h_1^{2s+1}\otimes r_0+u_1^{s+1}h_1^{2s}\otimes r_0\\ &= u_2^{s-1}h_1\otimes u_2 r_0+u_1^s h_1^{2s+1}\otimes r_0+u_1^{s+1}h_1^{2s}\otimes r_0\,. \end{aligned}$$

The first term is zero because $|h_1|\leqslant|u_2r_0|$. The last is zero by the fact that $u_1^s h_1^{2s}\otimes r_0=0$. $\square$

Lemma 2.7. *As a left BP_* module, W is generated by all $h^J\otimes r_c$, $l(J)>c\geqslant 0$, $j_1\in Z$. The relations among these elements are given by all*

$$h^J\otimes u_i r_c,\ 0\leqslant l(J)-c<2^i \quad \textit{and} \quad h^J\otimes\sum_{i=0}^{c} a_i r_{c-i}\,. \qquad \square$$

Proof. This follows from 1.6 and 2.1. $\square$

Definition 2.8. Let W' be the BP_* module with generators and relations as in 2.7 except do not use the relations $h^J\otimes u_1 r_c$, $l(J)=c$.

We have a surjection $W'\to W$. We believe that the elements $h^J\otimes u_1 r_c=$

$u_1(h^J \otimes r_c) + h^{J+\Delta_1} \otimes 2r_c = h^{J+\Delta_1} \otimes 2r_c$, $l(J) = c$, are already zero in W' and therefore that $W' \simeq W$. If this is true, the Bendersky-Davis conjectures would follow from the next proposition because it shows that $u_2^s h_1 \otimes r_0$ would be nonzero (then use 2.6).

Proposition 2.9. *There is a filtration on W' with associated graded object $E_0 W'$ such that*

(a) $(2, u_1)E_0 W' = 0$, and

(b) *$E_0 W'$ is a free $BP_*/(2, u_1)$ module on generators $h^J \otimes r_c$, $l(J) > c \geqslant 0$.* □

Remark. Since $W' \to W$ is surjective we have a similar filtration for W and (a) holds.

Proof. Let W_1 be the free left BP_* module on $h^J \otimes r_c$, $l(J) > c \geqslant 0$. Define a filtration by: $F_c W_1$ is the submodule generated by all $h^J \otimes r_i$, $i < c$. The element $h^J \otimes \sum a_i r_{c-i}$ is represented by $2(h^J \otimes r_c)$ in $E_0 W_1$. Let W_2 be W_1 after setting the elements $h^J \otimes \sum a_i r_{c-i}$ equal to zero. We have a filtration $F_c W_2$. This shows that $(2)E_0 W_2 = 0$ and $E_0 W_2$ is free over $BP_*/(2)$ on the $h^J \otimes r_c$, $l(J) > c \geqslant 0$. We filter $E_0 W_2$ by letting $F_J E_0 W_2$ be the submodule generated by $h^{J'} \otimes r_c$, $J' < J$. From 2.7, the elements used to obtain W' from W_2 are the $h^J \otimes u_i r_c$, $0 \leqslant l(J) - c < 2^i$, $i > 1$, $l(J) = c+1$, $i = 1$. They live in W_2. Their representatives in $E_0 E_0 W_2$ are given $u_1 h^{J+(2^i-2)\Delta_1} \otimes r_c$, $0 \leqslant l(J) - c < 2^i$, $i > 1$, $u_1 h^J \otimes r_c$, $l(J) = c+1$. This follows from 1.8 and $2E_0 W_2 = 0$. Reindexing, these elements are just $u_1 h^J \otimes r_c$, $l(J) > c$. Using these relations to define W' from W_2 the results, both (a) and (b), follow. (We would have some redundancy if we tried to use $h^J \otimes u_1 r_c$, $l(J) = c$ as well, and (b) would not follow.) □

Our attention is now fixed on the elements $2h^J \otimes r_c$, $l(J) = c+1$, because from the discussion before 2.9 we have:

Corollary 2.10. *If, for all $l(J) = c+1$, $0 = 2(h^J \otimes r_c) \in W'$, then $W' \simeq W$ and conjecture 7 is true.* □

We now throw away all of the elements which computations have taught us are irrelevant, just as Bendersky and Davis did. Let

$$\tilde{W}' \equiv BP\langle 2\rangle_* \otimes_{BP_*} W'/(h_2, h_3, \cdots) \tag{2.11}$$
$$\tilde{W} \equiv BP\langle 2\rangle_* \otimes_{BP_*} W/(h_2, h_3, \cdots)\,.$$

The following is an automatic Corollary of 2.7, 2.9 and 2.10.

Corollary 2.12. (a) *As a* $BP\langle 2\rangle_*$ *module,* $\tilde{W}$ *is generated by* $h_1^j\otimes\gamma_c$, $j>c\geqslant 0$ *with relations* $h_1^c\otimes u_i\gamma_c$, $0\leqslant j-c<2^i$ *and* $h^j\otimes\sum_i a_i\gamma_{c-i}$.

(b) $\tilde{W}'$ *is as in* (a) *but without the relations* $h_1^c\otimes u_1\gamma_c=2h_1^{c+1}\otimes\gamma_c$.

(c) *There is a filtration on* $\tilde{W}'$ *such that* $(2, u_1)E_0\tilde{W}'=0$, *and*

(d) $E_0\tilde{W}'$ *is a free* $Z/2[u_2]$ *module on* $h_1^j\otimes\gamma_c, j>c\geqslant 0$.

(e) *If* $2h_1^{c+1}\otimes\gamma_c=0$, $c\geqslant 0$, *in* $\tilde{W}'$, *then* $\tilde{W}'\simeq\tilde{W}$ *and Conjecture* 7 *is true.* □

More is actually true, because if $2h_1^{c+1}\otimes\gamma_c=0$, then $2h_1^c\otimes\gamma_{c-1}=0$ in $\tilde{W}'$. We prove a stronger result.

Define $\tilde{W}(k)=\tilde{W}'/(2h_1^{c+1}\otimes\gamma_c,\ c<k)$. Then $\tilde{W}(k)\simeq\tilde{W}$ for degrees less than $|2h_1^{k+1}\otimes\gamma_k|$.

Proposition 2.13. *If* $2h_1^{3k+1}\otimes\gamma_{3k}=0$ *in* $\tilde{W}(3k)$, *then*

(a) $2h_1^{c+1}\otimes\gamma_c=0$ *in* $\tilde{W}'$, $c\leqslant 3k+2$,

(b) $\tilde{W}'\simeq\tilde{W}$ *in degrees less than* $12(k+1)+2$, *and*

(c) *Conjecture* 7(*s*) *is true for* $s\leqslant 2k+1$. □

Remark 2.14. Hidden in this proposition is the fact that if we prove 7(*s*) for $2k$ using our approach then it is true for $2k+1$ as well, thus our $s=6$ implies $s=7$.

Proof. We define

$$\text{(2.15)}\qquad \beta\colon \Sigma^{-4}W(k)\to W(k-1)\,,\quad \beta(h^J\otimes\gamma_c)=h^{J-\Delta_1}\otimes\gamma_{c-1}\,,\quad \gamma_{-1}=0\,.$$

It is easy to verify that this is well defined. The map β can also be defined on $\tilde{W}'$, $\tilde{W}$, W' and W.

Claim. $2h_1^{k+1}\otimes\gamma_k=0$ in $\tilde{W}(k)$ implies $2h_1^{c+1}\otimes\gamma_c=0$ in $\tilde{W}'$, $c\leqslant k$.

Proof of claim. We use induction on k. If $2h_1^{k+1}\otimes\gamma_k=0$ in $\tilde{W}(k)$, apply β to deduce that $2h_1^k\otimes\gamma_{k-1}=0$ in $\tilde{W}(k-1)$. By induction we have $2h_1^{i+1}\otimes\gamma_i=0$ in $\tilde{W}'$, $i<k$. But now $\tilde{W}(k)\simeq\tilde{W}'$, so $2h_1^{k+1}\otimes\gamma_k=0$ in $\tilde{W}'$ as well.

Change k to $3k$ and we have proven (a) for $c\leqslant 3k$. We define another map

$$\text{(2.16)}\qquad \alpha\colon \Sigma^{-2}\tilde{W}'\to\tilde{W}'\,,\quad \alpha(h^J\otimes\gamma_c)=h^{J-\Delta_0}\otimes\gamma_c\,.$$

It is easy to show that this is well defined and exists on $\tilde{W}$, W', and W also.

We consider $2h_1^{3k+2}\otimes\gamma_{3k+1}$. By 2.12 (d),

$$2h_1^{3k+2}\otimes r_{3k+1}=\sum_{i,j} a_{ij}u_2^{k_{ij}}h_1^{i+j}\otimes r_i\,,$$

uniquely with $a_{ij}=0$ or 1.

Apply α to both sides. The left side is zero, so the only possible nonzero coefficients on the right are a_{i1}, so

$$2h_1^{3k+2}\otimes r_{3k+1}=\sum_{i\geqslant 0} a_iu_2^{k_i}h_1^{i+1}\otimes r_i\,.$$

Now apply β. We already have that $2h_1^{3k+1}\otimes r_{3k}=0$, so the only a_i that can be nonzero is a_0, so

$$2h_1^{3k+2}\otimes r_{3k+1}=a_0u_2^{k_0}h_1\otimes r_0\,.$$

Comparing the degree of the left hand side with the right we have $12k+6\neq 6k_0+2$, so $a_0=0$. The proof for $2h_1^{3k+3}\otimes r_{3k+2}=0$ is the same.

Part (b) follows. Part (c) follows from (b). □

Proof of Theorem 13. Define $B(k)/R\equiv B/(R+(x^{3k+2}))$, map

$$B(k)/R\to\Sigma^{-(12k+4)}\tilde{W}(3k)$$

by $h_1^j\otimes x^i\to h_1^{3k+1+j}\otimes r_{3k+1-i}$.

It is easy to check that the map is well defined. Furthermore, the map is an isomorphism in negative degrees. This is true simply because the generators and relations go to the generators and relations. It may be helpful, however, to observe first that

$$\widetilde{BP_*[[x]]/([2](x),\,x^{n+1})}\simeq\Sigma^{-2n-3}\widetilde{BP}_*RP^{2n+1}\,,$$

just from generators and relations.

The map takes $1\otimes x$ to $h_1^{3k+1}\otimes r_{3k}$, so if $2(1\otimes x)=0$, we can apply Proposition 2.13 (c) to prove part (b). Part (a) follows from (b). Part (c) is gruesome computation actually carried out modulo x^{11}, but the proofs of 2.13 and Theorem 13 imply that it is true modulo x^{13} also. □

The main differences in our approach to proving Conjecture 7 and that in **[B-D]** are threefold. We like to eliminate powers of 2. They like to eliminate powers of u_1. We like the left BP_* module structure, they like to move things to the right. We use Boardman's basis. Bendersky and Davis told us before we began this project that the [2]-sequence played a nasty roll. The only way to approach Conjecture 12 is to write $2(1\otimes x)=-1\otimes\sum_{i>0} a_ix^{i+1}$, and it seems that we have reduced the entire problem to the nasty [2]-sequence.

The only positive, non computational step we made towards a proof of Conjecture 12 is the following.

Proposition 2.17. *In B/R,*

$$2(1\otimes x) = u_1^2h_1^{-1}\otimes x^2 +_R u_2^2 h_1^{-3}\otimes x^4 . \qquad \square$$

Remarks 2.18. The term $u_1^2h_1^{-1}\otimes x^2$ can easily be replaced with $h_1\otimes(2x)^2$, and the [2]-sequence rears its ugly head again. If we do not tensor with $BP\langle 2\rangle_*$, this is

$$2(1\otimes x) = \sum_{i>0}^R u_i^2 h_1^{1-2^i}\otimes x^{2^i} .$$

Proof. Set $h_i=0$, $i>1$ and subtract the equal terms $u_0 y$ and $\eta(u_0)y$; 1.2 becomes

$$\sum_{i>0}^R u_i y^{2^i} +_R \sum_{i\geqslant 0}^R u_i^2 h_1 y^{2^{i+1}} = \sum_{i>0}^R \eta(u_i) y^{2^i} +_R \sum_{i\geqslant 0}^R h_1^{2^i} u_i y^{2^{i+1}} .$$

The dummy variable has degree -2. Substitute $y=(h_1^{-1}\otimes x)^{1/2}$ and we have a formula in B/R;

$$\sum_{i>0}^R u_i h_1^{-2^{i-1}}\otimes x^{2^{i-1}} +_R \sum_{i\geqslant 0}^R u_i^2 h_1^{1-2^i}\otimes x^{2^i} = \sum_{i>0}^R h_1^{-2^{i-1}}\otimes u_i x^{2^{i-1}} +_R \sum_{i\geqslant 0}^R 1\otimes u_i x^{2^i} .$$

The far right sum is $1\otimes\sum_{i\geqslant 0}^F u_i x^{2^i}=1\otimes[2](x)=0$. The terms of the other sum on the right have $|h_1^{-2^{i-1}}|\leqslant|u_i x^{2^{i-1}}|$ so they are all zero except $h_1^{-1}\otimes u_1 x$, which only has one power of x so it is not a given relation. The same thing occurs in the very first sum. The first term of the second sum is $4(1\otimes x)$. Using **[B-D]** it is easy to see that this is zero. We have

$$4(1\otimes x) = 2(1\otimes 2x) = -2(1\otimes \sum_{i>0} a_i x^{i+1}) .$$

Now, $2=-u_1h_1^{-1}+h_1^{-1}u_1$, so this is

$$+u_1h_1^{-1}\otimes\sum_{i>0} a_i x^{i+1} - h_1^{-1}\otimes u_1 \sum_{i>0} a_i x^{i+1} = 0 ,$$

because we never use only one power of x. There is some checking to do to see that the formal group sums do not alter these zeros. We have

$$\sum_{i>0}^R u_i^2 h_1^{1-2^i}\otimes x^{2^i} = h_1^{-1}\otimes u_1 x -_R u_1 h_1^{-1}\otimes x .$$

$$= \sum_{i,j} u_1^j h_1^{-i-j}\otimes u_1^i x^{i+j} a_{ij}^1, \quad \text{some} \quad a_{ij}^1\in BP_{2(i+j-1)} .$$

Again, all terms are zero except $h_1^{-1}\otimes u_1 x - u_1 h_1^{-1}\otimes x$ which is $u_1h_1^{-1}\otimes x + 2(1\otimes x) - u_1 h_1^{-1}\otimes x = 2(1\otimes x)$. $\square$

A mindless algorithm for computing with Conjecture 12 is as follows. (1) Take the lowest power of x with a non-zero coefficient. (2) Use the right unit to move u's to the left. (3) Eliminate 2 from the coefficient by use of the [2]-sequence. (4) For a term with u_1, say $au_1h_1^i\otimes x^j$, $a\in BP\langle 2\rangle_*$, we must have $i>j$ or it is zero. Find the smallest k such that $|h_1^{i-(2^k-2)}|\leqslant u_kx^j$. Then add $ah_1^{i-(2^k-2)}\otimes u_kx^j=0$ by using the right unit. From 1.8 we see that this will get rid of the term with the u_1. This is all of the algorithm, just repeat it. The reason this is all, is because, so far, we are never left with a term $u_2^kh_1^i\otimes\gamma_j$. If this ever happens then our concept of W is all wrong. Moreover, we have shown that if it ever happens, the first case must be $u_2^{2k}h_1^{-3k}\otimes x^{3k+1}$, for some k. Computations can be shortened significantly by using the "excess" in **[B-D]**. For example, to start, we take $2(1\otimes x)=-\sum_{i>0}1\otimes a_ix^{i+1}$. However, $2\,|\,a_{2k}$, so all terms $1\otimes a_{2k}x^{2k+1}=0$ automatically.

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Note added in proofs. Conjecture 1 has recently been proved by Davis using Astey's direct approach.