

On the Hopf ring for the sphere

Peter J. Eccles, Paul R. Turner, W. Stephen Wilson

¹ Mathematics Department, Manchester University, Manchester M13 9PL, England
(e-mail: peter@ma.man.ac.uk)

² Mathematisches Institut der Universität, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany
(e-mail: pault@vogon.mathi.uni-heidelberg.de)

³ Mathematics Department, Johns Hopkins University, Baltimore, Maryland 21218, USA
(e-mail: wsw@math.jhu.edu)

Received 17 March 1994; in final form 1 April 1995

1 Introduction

Given any ring spectrum G there are infinite loop spaces \mathbf{G}_k representing the cohomology functor $G^k(-)$. To study the (mod 2) homology of these spaces it is convenient to consider the bigraded object $\{H_*\mathbf{G}_k\}_{k \in \mathbb{Z}}$. Each $H_*\mathbf{G}_k$ is a Hopf algebra and if G is a ring spectrum we are fortunate enough to have another product

$$\circ : H_*\mathbf{G}_k \otimes H_*\mathbf{G}_l \rightarrow H_*\mathbf{G}_{k+l}$$

at our disposal. This leads us to consider $\{H_*\mathbf{G}_k\}_{k \in \mathbb{Z}}$ as a ring object in the category of graded coalgebras i.e., as a Hopf ring. We refer the reader to [4] for the appropriate background details and discussion.

One philosophy on how to approach Hopf ring calculations, advocated by the third author for some time, is that if both π_*G and H_*G are known then with a bit of luck the Hopf ring $H_*\mathbf{G}_*$ can be computed. This approach has worked well with a number of important examples, particularly for complex oriented spectra. In cases when π_*G is unknown, a different approach must be adopted. In general it is too much to ask for a complete computation of $H_*\mathbf{G}_*$ as this would imply a knowledge of π_*G , occurring as the ring–ring $H_0\mathbf{G}_* \cong \mathbb{F}_2[G_*]$. It may however be possible to identify several important sub Hopf rings, of which $H_*\mathbf{G}_0$ and $\{H_*\mathbf{G}_k\}_{k \geq 0}$ are of particular interest. In this paper we are concerned with the case when $G = S$, the sphere spectrum, and where \mathbf{S}_k is more usually denoted by $QS^k = \lim_{n \rightarrow \infty} \Omega^n S^{n+k}$. The homology of these spaces has been extensively studied (see for example [3], [2] and [1]) and is by now well understood. Our contribution is a recasting of the results into the framework of Hopf rings and thereby making very explicit the relationship between the two available products. We build on [6], where the mod 2 Hopf ring structure of H_*QS^0 was established, and compute the mod 2 homology Hopf ring $\{H_*QS^k\}_{k \geq 0}$, a sub Hopf ring of the full Hopf ring associated to the sphere spectrum. We also produce new algebra generators for H_*QS^k when

$k \geq 0$. We adopt the Hopf ring conventions established in [4] with one notable exception: we denote star product by juxtaposition, omitting the traditional $*$ symbol. Throughout this paper we are only concerned with mod 2 homology and we write H_*X for $H_*(X; \mathbb{F}_2)$. This paper is the natural sequel to [6] and a certain familiarity with the results therein is assumed.

The second author expresses thanks to the Alexander von Humboldt Stiftung for providing financial support during the period when this note was written up.

2 Statement of results

Typically homology Hopf rings have few generators and many relations, often presented in terms of formal power series. We begin by identifying a small set of generators for $\{H_*QS^k\}_{k \geq 0}$. Recall that $[1] \in H_0QS^0$ is the image of the non base-point generator under the inclusion $S^0 \hookrightarrow QS^0$. Recall also that $[-1]$ denotes $\chi([1])$, where χ is the antipode of H_*QS^0 induced by the map reversing loop direction. Let $E_k = Q^k[1] \in H_kQS^0$, where Q^k is the k^{th} Dyer–Lashof operation, and let $e \in H_1QS^1$ be the image of the generator under the inclusion $S^1 \hookrightarrow QS^1$. These elements will be the generators of our Hopf ring. Note that the element e , which is usually known as the suspension element since $e \circ x$ is the homology suspension of x , is primitive and that the coproduct on the E_k is given by comparing coefficients in the formal power series equality

$$\psi(E(s)) = E(s) \otimes E(s)$$

where $E(s)$ is defined to be the formal power series $\sum_{k \geq 0} E_k s^k$. To proceed we must establish some relations between the above classes. The relations we require turn out to have a remarkably simple description.

Proposition 2.1 *In the formal power series ring $\{H_*QS^k\}_{k \geq 0}[s, t]$ we have the following relations.*

$$(2.2) \quad E_0 = [2]$$

$$(2.3) \quad E(s) \circ E(t) = E(s) \circ E(s + t)$$

$$(2.4) \quad e^{oi} \circ E_i = (e^{oi})^2$$

Proof. Equations (2.2) and (2.3) are statements about H_*QS^0 and were proved in [6]. For (2.4) it is standard that

$$(2.5) \quad Q^i(e^{om} \circ a) = e^{om} \circ Q^i(a)$$

for $a \in \{H_*QS^k\}_{k \geq 0}$ so we have

$$\begin{aligned} e^{oi} \circ E_i &= e^{oi} \circ Q^i[1] \\ &= Q^i(e^{oi} \circ [1]) \quad \text{by (2.5)} \\ &= Q^i(e^{oi}) \\ &= (e^{oi})^2 \quad \text{by dimensions.} \quad \square \end{aligned}$$

With this minimal preparation we can now state our main results. Let $D_{*,0}$ be the graded \mathbb{F}_2 coalgebra on generators $E_i \in D_{i,0}, i \geq 0$ with $\psi(E_i) = \sum_{k+l=i} E_k \otimes E_l$. Let $D_{*,1}$ be the graded \mathbb{F}_2 coalgebra on the single primitive generator $e \in D_{1,1}$ and for $k \geq 2$ let $D_{*,k}$ be the graded \mathbb{F}_2 coalgebra with $D_{0,k} = \mathbb{F}_2$ and zero in other dimensions. To construct a Hopf ring out of the coalgebra $D_{*,*}$ we must appeal to the free Hopf ring functor of Ravenel and Wilson [4].

Theorem 2.6 *Let $A_{*,*}$ be the free $\mathbb{F}_2[\mathbb{Z}]$ -Hopf ring on $D_{*,*}$ modulo all the relations implied by Proposition 2.1. There is an isomorphism of Hopf rings*

$$A_{*,*} \cong \{H_*QS^k\}_{k \geq 0}.$$

Theorem 2.7 *Space by space we have the following Hopf algebras.*

$$H_*QS^0 = P[E_{i_0} \circ E_{2(i_0+i_1)} \circ \cdots \circ E_{2^{n-1}(i_0+\cdots+i_{n-1})} | n \geq 1, i_0 > 0] \otimes \mathbb{F}_2[\mathbb{Z}],$$

and for $k > 0$ we have

$$H_*QS^k = P[e^{\circ k} \circ E_{i_0} \circ E_{2(i_0+i_1)} \circ \cdots \circ E_{2^{n-1}(i_0+\cdots+i_{n-1})} | n \geq 1, i_0 > k] \otimes \mathbb{F}_2[e^{\circ k}].$$

The coproduct is specified by

$$\psi(e) = 1 \otimes e + e \otimes 1 \quad \psi(E(s)) = E(s) \otimes E(s) \quad \psi(a \circ b) = \psi(a) \circ \psi(b).$$

Remark 2.8 The action of the mod 2 Dyer–Lashof algebra on the above Hopf ring is completely described by Theorem 5.1 in [6] together with the additional formula $Q^i(e^{\circ k} \circ a) = e^{\circ k} \circ Q^i(a)$.

3 Proofs

Since the defining relations of $A_{*,*}$ hold in $\{H_*QS^k\}_{k \geq 0}$ by Proposition 2.1, there is a map of Hopf rings $A_{*,*} \rightarrow \{H_*QS^k\}_{k \geq 0}$. Hence Theorem 2.6 follows from Theorem 2.7 by simply using the Hopf ring structure and the relations in Proposition 2.1. To prove Theorem 2.7 we use an inductive argument based on the bar spectral sequence. Recall that in the context of Hopf rings the bar spectral sequence appears in the form

$$(3.1) \quad E_{**}^2 = \text{Tor}_{**}^{F_*G_k}(F_*, F_*) \Rightarrow F_*\mathbf{G}'_{k+1}$$

for spectra F and G , and comes equipped with a pairing [5]

$$(3.2) \quad \circ : E_{p,*}^r(F_*\mathbf{G}_j) \otimes_{F_*} F_*\mathbf{G}_k \rightarrow E_{p,*}^r(F_*\mathbf{G}_{j+k})$$

with

$$d^r(x) \circ y = d^r(x \circ y)$$

converging to the usual circle product at E^∞ .

Our induction is grounded by the description of H_*QS^0 given in [6]. We assume H_*QS^k is as described in the theorem and we prove the result for H_*QS^{k+1} .

According to (3.1) the bar spectral sequence has the form

$$E^2 = \text{Tor}^{H_*QS^k}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*QS^{k+1} .$$

It is standard that Tor of a polynomial algebra is an exterior algebra on the suspension of the polynomial generators, giving

$$\begin{aligned} \text{Tor}^{H_*QS^k}(\mathbb{F}_2, \mathbb{F}_2) &= E[e^{\circ k+1} \circ E_{i_0} \circ \cdots \circ E_{2^{n-1}(i_0+\cdots+i_{n-1})} | n \geq 1, i_0 > k] \\ &\quad \otimes \mathbb{F}_2[e^{\circ k+1}] . \end{aligned}$$

The spectral sequence is easily seen to collapse by the pairing (3.2). To prove the result we must solve the extension problems, turning our presently exterior generators into polynomial ones and ridding ourselves of all those potential generators beginning $e^{\circ k+1} \circ E_{k+1}$. By a standard Hopf ring manipulation we have

$$\begin{aligned} (3.3) \quad &(e^{\circ k+1} \circ E_{i_0} \circ E_{2(i_0+i_1)} \circ \cdots \circ E_{2^{n-1}(i_0+\cdots+i_{n-1})})^2 \\ &= (e^{\circ k+1})^2 \circ E_{2i_0} \circ E_{4(i_0+i_1)} \circ \cdots \circ E_{2^n(i_0+\cdots+i_{n-1})} \\ &= e^{\circ k+1} \circ E_{k+1} \circ E_{2i_0} \circ E_{4(i_0+i_1)} \circ \cdots \circ E_{2^n(i_0+\cdots+i_{n-1})} \quad \text{by (2.4)} . \end{aligned}$$

Since $i_0 > k$ this is of the form of one of our exterior generators. In particular this proves that all powers of each of the elements $e^{\circ k+1} \circ E_{i_0} \circ E_{2(i_0+i_1)} \circ \cdots \circ E_{2^{n-1}(i_0+\cdots+i_{n-1})}$ are non-zero and independent so we have polynomial generators as required. Further, (3.3) shows that any element of the form

$$e^{\circ k+1} \circ E_{k+1} \circ E_{2(k+1+i_0)} \circ \cdots \circ E_{2^n(k+1+i_0+\cdots+i_{n-1})}$$

may be rewritten as the square

$$(e^{\circ k+1} \circ E_{k+1+i_0} \circ \cdots \circ E_{2^{n-1}(k+1+i_0+\cdots+i_{n-1})})^2$$

and so cannot be a generator. So we have also succeeded in eliminating everything beginning $e^{\circ k+1} \circ E_{k+1}$ and we are left with the claimed polynomial generators.

References

1. Fred R. Cohen, T.J. Lada, J. Peter May, The Homology of Iterated Loop Spaces. Lecture Notes in Mathematics **533** (Springer Verlag 1976)
2. Eldon Dyer, Richard K. Lashoff, Homology of iterated loop spaces. American Journal of Mathematics **84**, 35–88 (1962)

3. T. Kudo, S. Araki, Topology of H_n -spaces and H -squaring operations. *Memoirs of the Faculty of Science* **10**(2), 85–120 (Kyusyu University 1956)
4. Douglas C. Ravenel, W. Stephen Wilson, The Hopf ring for complex cobordism. *Journal of Pure and Applied Algebra* **9**, 241–280 (1977)
5. Robert W. Thomason, W. Stephen Wilson, Hopf rings in the bar spectral sequence. *Quarterly Journal of Mathematics Oxford* **31**, 507–511 (1980)
6. Paul R. Turner, Dickson coinvariants and the homology of QS^0 . This volume