

# Morava Hopf algebras and spaces $K(n)$ equivalent to finite Postnikov systems

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## Abstract

We have three somewhat independent sets of results. Our first results are a mixed blessing. We show that Morava  $K$ -theories don't see  $k$ -invariants for homotopy commutative  $H$ -spaces which are finite Postnikov systems, i.e. for those with only a finite number of homotopy groups. Since  $k$ -invariants are what holds the space together, this suggests that Morava  $K$ -theories will not be of much use around such spaces. On the other hand, this gives us the Morava  $K$ -theory of a wide class of spaces which is bound to be useful. In particular, this work allows the recent work in [RWY] to be applied to compute the Brown-Peterson cohomology of all such spaces. Their Brown-Peterson cohomology turns out to be all in even degrees (as is their Morava  $K$ -theory) and flat as a  $BP^*$  module for the category of finitely presented

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$BP^*(BP)$  modules. Thus these examples have extremely nice Brown-Peterson cohomology which is as good as a Hopf algebra.

Our second set of results produces a large family of spaces which behave as if they were finite Postnikov systems from the point of view of Morava  $K$ -theory even though they are not. This allows us to apply the above results to an even wider class of spaces than finite Postnikov systems. These examples come from spaces in omega spectra with certain properties. There are many well known examples with these properties. In particular, we compute the  $K(n)$  homology of all the spaces in the  $\Omega$ -spectra for  $P(q)$  and  $k(q)$  where  $q > n$ .

In order to prove our results on finite Postnikov systems we need our third set of results; a beginning of an analysis of bicommutative Hopf algebras over  $K(n)_*$ .

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## 1 Introduction

The Morava  $K$ -theories are a collection of generalized homology theories which are intimately connected to complex cobordism ([JW75], [Wür91]). It is known that they play a central role in aspects of homotopy theory ([Hop87],[DHS88],[HS]). This and their relative computability (due to a Künneth isomorphism) makes them a powerful tool. For each prime,  $p$ , and  $n > 0$ , the coefficient ring for  $K(n)_*(-)$  is  $K(n)_* \simeq \mathbf{F}_p[v_n, v_n^{-1}]$  where the degree of  $v_n$  is  $2(p^n - 1)$ .

We assume that all of our spaces are homotopy equivalent to a CW complex.

**Theorem 1.1** *Let  $X$  be a connected  $p$ -local space with  $\pi_k(X)$  finitely generated over  $\mathbf{Z}_{(p)}$ ,  $k > 1$ , and non-zero for only a finite number of  $k$ . Then  $K(n)_*(\Omega X)$  has a natural filtration by normal sub-Hopf algebras:*

$$K(n)_* \simeq F_{-(n+2)} \subset \cdots \subset F_{-2} \subset F_{-1} \subset F_0 \simeq K(n)_*(\Omega X)$$

with

$$F_{-q} // F_{-(q+1)} \simeq K(n)_*(K(\pi_q(\Omega X), q))$$

as Hopf algebras.

If  $K(n)_*(\Omega X)$  is commutative (e.g. if  $X$  is an  $H$ -space), then  $K(n)_*(\Omega X)$  is isomorphic, as a Hopf algebra, to the associated graded object above:

$$K(n)_*(\Omega X) \simeq \bigotimes_{0 \leq m \leq n+1} K(n)_*(K(\pi_m(\Omega X), m)).$$

This is natural if either all  $\pi_k(X)$ ,  $k > 1$ , are finite, or if they are all free.

We believe that if  $K(n)_*(X)$  is not commutative then we still have the last isomorphism as coalgebras but have been unable to prove it. It may require a dual Borel theorem for our Hopf algebras and we have only been able to handle the bicommutative case.

We actually state and prove a theorem (Theorem 2.1) which does not require the homotopy groups to be finitely generated over  $\mathbf{Z}_{(p)}$ . We can have copies of  $\mathbf{Q}/\mathbf{Z}_{(p)}$  but our groups can only have a finite number of summands. This is important both for our proofs and for some of our applications. The last naturality statement is true in this case if all the homotopy groups are torsion.

Note also that the  $n = 0$  case of this theorem is a familiar result about rational homology.

In [MM92], McCleary-McLaughlin show that for an Eilenberg–Mac Lane space  $X$  with finite homotopy group,  $K(n)_*(X)$  has the same rank as

$K(n-1)_*(\mathcal{L}X)$ , where  $\mathcal{L}X$  denotes the free loop space of  $X$ . In view of the theorem above, the following generalization of their result is immediate.

**Corollary 1.2** *Let  $X$  be a simply connected  $H$ -space with finitely many non-trivial homotopy groups, each of which is finite. Then  $K(n)_*(X)$  has the same rank as  $K(n-1)_*(\mathcal{L}X)$ .*

The following corollary follows from the fact that the Morava  $K$ -theory of Eilenberg–Mac Lane spaces is even degree (except for the circle) and from the main results of [RWY]. The condition on  $\pi_1$  is needed to avoid having copies of the circle in our space, giving us odd degree elements.

**Corollary 1.3** *If  $X$  is as in Theorem 1.1 and  $\pi_1(\Omega X)$  is torsion, then  $K(n)_*(\Omega X)$  is even degree and so is  $BP_p^{**}(\Omega X)$  where  $BP_p^\wedge$  is the  $p$ -adic completion of  $BP$ . If  $\pi_k(X)$  is finite for  $k > 1$  then  $BP^*(\Omega X)$  is even degree. In either case, it is a flat  $BP^*$ -module for the category of finitely presented  $BP^*(BP)$ -modules.*

We see the Morava  $K$ -theory cannot distinguish between the double loops of such a space and a product of Eilenberg–Mac Lane spaces with the same homotopy groups. One cannot expect to generalize this too much to spaces with an infinite number of non-zero  $k$ -invariants; the sphere,  $S^k$ , is a counter example to that. Inverse limit problems rear their ugly head. Morava  $K$ -theory somehow looks at the whole space rather than how it is put together. On the other hand, our result certainly does cover spaces with infinitely many homotopy groups if the  $k$ -invariants are zero for all but a finite number of stages. This is because the Eilenberg–Mac Lane spaces split off as a product if the  $k$ -invariant is zero. Another, more substantial direction of generalization is to spaces in  $\Omega$ -spectra with certain stable properties.

Although the abstract isomorphism is interesting from a theoretical point of view, the practical value comes because the Morava  $K$ -theory of Eilenberg–Mac Lane spaces is completely known, [RW80]. In particular, it is always even

degree (except for the circle). We also know that

$$K(n)_*(K(\pi_m(\Omega X), m)) \simeq K(n)_*$$

if  $m > n + 1$ , so that Morava  $K$ -theory does not see the higher homotopy groups of these spaces, and spaces with only higher homotopy groups are acyclic, e.g. if  $\Omega X$  is  $n$  connected and  $\pi_{n+1}(\Omega X)$  is torsion. When we have acyclicity we do not need Hopf algebras much:

**Theorem 1.4** *Let  $X$  an  $n$ -connected  $p$ -local space with  $\pi_k(X)$  a finitely generated  $\mathbf{Z}_{(p)}$  module which is non-trivial for only a finite number of  $k$ . If  $\pi_{n+1}(X)$  is torsion, then*

$$K(n)_*(X) \simeq K(n)_*.$$

This theorem is rather easy and is proven quite quickly directly from [RW80].

The proof of Theorem 1.1 is given in Section 2 modulo certain general results about graded Hopf algebras over  $K(n)_*$  which will be proven in Section 4. It mimics a proof for rational homology. The only place where Section 4 is used is in showing the final statement of Theorem 1.1 about the splitting as Hopf algebras.

In Section 3, which is independent of the rest of the paper, we will prove some results about spaces in the  $\Omega$ -spectrum of a  $K(n)_*$ -acyclic spectrum  $X$ . Recall that an  $\Omega$ -spectrum  $X = \{\underline{X}_i\}$  has  $\Omega \underline{X}_{i+1} = \underline{X}_i$ . A motivating example for this study was produced by Richard Kramer's work computing  $K(n)_*(\underline{k}(q)_*)$  when  $n < q$ .

**Theorem 1.5** *Let  $X = \{\underline{X}_i\}$  be a connective  $\Omega$ -spectrum of finite type with bottom cell in dimension 0 and  $K(n)_*(\underline{X}_m) \simeq K(n)_*$  for some  $m$ . Let  $X \rightarrow F$  be a map to a finite Postnikov system which is an equivalence through dimension  $n + 1$ . Then  $\underline{X}_q$  is  $K(n)_*$ -equivalent to  $\underline{F}_q$  for all  $q \geq 0$ .*

This is Theorem 3.7. We can now apply Theorem 1.1 to such a spectrum to get:

**Theorem 1.6** *Let  $X$  be as in Theorem 1.5, then, for all  $k \in \mathbf{Z}$ ,*

$$K(n)_*(\underline{X}_k) \simeq \bigotimes_{n+1 \geq i \geq 0} K(n)_*(K(\pi_i(\underline{X}_k), i))$$

*as Hopf algebras. In particular, if  $k > n + 1$ , then  $K(n)_*(\underline{X}_k) \simeq K(n)_*$ . Also, if  $\pi_0(X)$  is torsion, then  $\underline{X}_{n+1}$  is  $K(n)_*$ -acyclic. Furthermore, whether  $K(n)_*(\underline{X}_k)$  is trivial or not for  $k \geq 0$  depends only on whether  $K(n)_*(K(\pi_0(X), k))$  is trivial or not.*

We call such a spectrum with one space  $K(n)_*$ -acyclic, *strongly  $K(n)_*$ -acyclic*, because it implies that almost all other spaces are also  $K(n)_*$ -acyclic. We can get the spaces  $\underline{X}_m$ ,  $m > n + 1$  to be  $K(n)_*$ -acyclic without reference to the first section. Note that being strongly  $K(n)_*$ -acyclic implies that the spectrum is  $K(n)_*$ -acyclic since  $K(n)_*(X) = \text{dir lim } K(n)_*(\underline{X}_k)$  using the suspension maps. Bousfield has a generalization of this which may have useful applications together with the rest of our work. We will discuss his results in Section 3.

This does not lead to a calculation of the Brown-Peterson cohomology as in the case of a real finite Postnikov system because it is easy to have spectra which are strongly  $K(n)_*$ -acyclic but have non-trivial  $K(n + 1)$  homology. However, following [RWY], it does lead to the calculation of the  $E(n)$  cohomology and a host of others.

What we need now is a condition on  $X$  which implies that it is strongly  $K(n)_*$ -acyclic. Associated with the “telescope” conjecture is a functor  $L_n^f$ , see [Rav93]. It supplies us with a class of examples. From Corollary 3.13 we have:

**Theorem 1.7** *A connective spectrum  $X$  for which  $L_n^f X$  is contractible is strongly  $K(n)_*$ -acyclic.*

In particular, a suspension spectrum of a finite complex which is  $K(n)_*$ -acyclic is strongly  $K(n)_*$ -acyclic. It is possible that the converse of this Theorem is also true. This result is not phrased in the most familiar or applicable of terms. When  $X$  is a  $BP$  module spectrum then this functor coincides with a more familiar one. Let  $E(n)$  be the homology theory with coefficient ring  $\mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$ . We have, from Theorem 3.14:

**Theorem 1.8** *Let  $X$  be a connective  $BP$  module spectrum which is  $E(n)_*(-)$ -acyclic, or, equivalently,  $K(q)_*(-)$ -acyclic for  $0 \leq q \leq n$ , then  $X$  is strongly  $K(n)_*$ -acyclic.*

What we need now are some concrete examples of interest. This result can be reduced, Corollary 3.15, to a simpler statement which we can use for this purpose.

**Corollary 1.9** *If  $X$  is a connective  $BP$ -module spectrum in which each element of  $\pi_*(X)$  is annihilated by some power of the ideal*

$$I_{n+1} = (p, v_1, v_2, \dots, v_n) \subset BP_*,$$

*then  $X$  is strongly  $K(n)_*$ -acyclic.*

Thus we see that any connective  $BP$  module spectrum with some power of  $I_{n+1}$  mapping to zero is strongly  $K(n)_*$ -acyclic. There are a lot of familiar examples in this collection. Recall that  $BP_* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \dots]$  where the degree of  $v_n$  is  $2(p^n - 1)$ . We have the spectra  $BP\langle n \rangle$ , with coefficient ring  $BP\langle n \rangle_* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n]$ , see [Wil75] and [JW73]. We also have  $P(k, n)$ , the spectrum with  $P(k, n)_* \simeq BP\langle n \rangle_* / I_k$  for  $0 \leq k \leq n$ . These theories are constructed using the usual Baas-Sullivan singularities [Baa73], and [BM71]. Many of these theories are already familiar. In particular,  $P(0, \infty) = BP$ ,  $P(n, n) = k(n)$ ,  $P(k, \infty) = P(k)$ , (see [JW75] and [Wür77]), and  $P(0, n) = BP\langle n \rangle$ . The spaces in the  $\Omega$ -spectra for  $P(k, n)$  are important in studying

the spaces in the  $\Omega$ -spectra for  $P(k)$  in [BW]. In addition, the theories  $E(k, n) = v_n^{-1}P(k, n)$ , play a prominent role in [RWY].

From the above theorem, we see:

**Corollary 1.10** *For  $m \geq q > n$ ,  $P(q, m)$  is strongly  $K(n)_*$ -acyclic, so, for  $k \in \mathbf{Z}$ ,*

$$K(n)_*(\underline{P(q, m)}_k) \simeq \bigotimes_{n+1 \geq i \geq 0} K(n)_*(K(\pi_i(\underline{P(q, m)}_k), i)).$$

We have explicitly computed the  $K(n)$  homology of all of the spaces in the  $\Omega$ -spectrum for all of these theories. Recall that this includes the more familiar  $P(q)$  and  $k(q)$  for  $q > n$ . The simplicity of the answer is in stark contrast with the calculation of  $K(n)_*(\underline{P(n)}_*)$  in [RW96] and of  $K(n)_*(\underline{k(n)}_*)$  in [Kra90].

The category of graded Hopf algebras over  $K(n)_*$  is equivalent to that of Hopf algebras over  $\overline{K(n)}_* \simeq \mathbf{F}_p$ , graded over  $\mathbf{Z}/(2p^n - 2)$  (where we have set  $v_n = 1$  in order to be working over a perfect field). Not everything about connected graded Hopf algebras carries over to these Hopf algebras so one must be somewhat careful. Being careful led us to initiate an investigation of the type of Hopf algebras that arise when you take the Morava  $K$ -theory of connected homotopy commutative  $H$ -spaces. The results of this investigation may well be of more interest than the applications to finite Postnikov systems and Section 4 is dedicated to this study.

There, we study what we call *commutative Morava homology Hopf algebras* (for a given  $n > 0$ ). This is the category,  $C(n)$ , of bicommutative, biassociative, Hopf algebras over  $\mathbf{F}_p$  which are graded over  $\mathbf{Z}/2(p^n - 1)$ . Furthermore, the primitive filtration is exhaustive and it is the direct limit of finite dimensional  $\mathbf{F}_p$  sub-coalgebras.  $\overline{K(n)}_*(\Omega X)$ , where  $X$  is an  $H$ -space, is such an object. We restrict our attention to such objects which are concentrated in even degrees. Denote this category by  $EC(n)$ . We show it is an abelian category. For the sake of completeness, we show that for odd primes

the bigger category,  $C(n)$ , splits into the product of  $EC(n)$  and  $OC(n)$ , where  $OC(n)$  consists of exterior algebras on odd degree primitive generators.

Since we deal only with evenly graded objects, our Morava homology Hopf algebras are really graded over  $G = \mathbf{Z}/(p^n - 1)$ . Let  $H$  be the cyclic group of order  $n$ .  $H$  acts on  $G$  via the  $p^{\text{th}}$  power map. Writing  $G$  additively, the map  $H \times G \rightarrow G$  is given by  $(i, j) \rightarrow p^i j$ . Let  $\gamma$  denote an  $H$ -orbit:

$$\gamma = \{j, pj, p^2j, \dots\} \subset G.$$

We have:

**Theorem 1.11** *For each evenly graded commutative Morava homology Hopf algebra, i.e. for  $A \in EC(n)$ , there is a natural splitting*

$$A \simeq \bigotimes_{\gamma} A_{\gamma}$$

where the tensor product is over all  $H$ -orbits  $\gamma$  and the primitives of  $A_{\gamma}$  all have dimensions in  $2\gamma$ .

To prove this theorem we construct idempotents in our category.

**Theorem 1.12** *For every  $A \in EC(n)$  there are canonical idempotents  $e_{\gamma}$  such that  $\sum_{\gamma} e_{\gamma} = 1_A$  and  $e_{\gamma}e_{\beta}$  is trivial if  $\gamma \neq \beta$ . These idempotents are natural. The idempotent  $e_{\gamma}$  sends all primitives to zero which do not have dimensions in  $2\gamma$  and is the identity on all primitives which are in dimensions in  $2\gamma$ .*

Theorem 1.11 follows immediately from this and the fact that tensor products are the sum in this category. Theorem 1.12, in turn, follows immediately from Theorem 1.13 which does even more for us. Let  $EC(n)_{\gamma}$  be the sub-category of  $EC(n)$  whose objects have primitives only in dimensions in  $2\gamma$ . What we really prove, in Section 4, is the following:

**Theorem 1.13** *There are commuting idempotent functors,  $e_\gamma$ , on  $EC(n)$  such that  $\sum_\gamma e_\gamma$  is naturally equivalent to the identity functor and  $e_\gamma e_\beta$  is the trivial functor if  $\gamma \neq \beta$ . As categories:*

$$EC(n) \simeq \prod_{\gamma} EC(n)_\gamma.$$

*In particular, there is only the trivial map  $e_\gamma(A) = A_\gamma \rightarrow e_\beta(B) = B_\beta$  if  $\gamma \neq \beta$ . Furthermore,  $EC(n)_\gamma$  is an abelian category.*

We can define a function  $\bar{\alpha}_p : \mathbf{Z}/(p^n - 1) \rightarrow \mathbf{Z}$  by just taking the usual lift to  $\mathbf{Z}$  and then the usual  $\alpha$ , the sum of the coefficients in the  $p$ -adic expansion of our number.  $\bar{\alpha}_p$  is constant on an orbit  $\gamma$ . Usually several orbits will have the same image under the map  $\bar{\alpha}_p$ . The following fact is a consequence of the computations done in [RW80], and is what we need to help prove Theorem 1.1.

**Theorem 1.14** *In the Hopf algebras  $\overline{K(n)}_*(K(T, q))$  for any abelian torsion group  $T$  and  $\overline{K(n)}_*(K(F, q+1))$  for any torsion free abelian group with  $q \geq 1$ , all orbits  $\gamma$  (as in 1.11) with nontrivial factors satisfy  $\bar{\alpha}_p(\gamma) = q$ .*

This is proved below as Theorem 4.14.

Our results were discovered while pursuing the *Johnson Question*, see [RW80, Section 13]. This assertion is that if  $0 \neq x \in BP_n(X)$  where  $X$  is a space, then  $x$  is not  $v_n$  torsion. This is a very strong unstable condition. At present, two of the authors have a good plausibility argument which they hope to turn into a proof some day. The approach which led to the present paper was just one of many dead ends. A cohomology theory can be defined by:

$$\mathrm{Hom}_{BP_*}(BP_*(X), Q/\mathbf{Z}_{(p)}).$$

The classifying space for the  $n$ -th group,  $\underline{M}_n$ , contains the universal example for the degree  $n$  *Johnson Question*. In an attempt to get some insight into its Brown-Peterson homology, the general phenomenon of Theorem 1.1 was

discovered. Each one of these spaces has only a finite number of non-trivial homotopy groups. Each non-trivial group is a finite sum of copies of  $Q/\mathbf{Z}_{(p)}$ . Theorem 2.1 still applies to it to give  $K(n)_*(\underline{M}_n)$ .

Morava  $K$ -theory can sometimes be problematic when  $p = 2$ . Because we are restricted to even degree objects this is not a problem for us, see Remark 4.4.

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## 2 Proof of the main theorem

For a group,  $G$ , let  $R[G]$  be the group ring for  $G$  over  $R$ . Let  $\tilde{X}$  be the universal cover for  $X$ . We have a sequence of fibrations, up to homotopy:

$$\Omega\tilde{X} \rightarrow \Omega X \rightarrow \pi_1(X) \rightarrow \tilde{X} \rightarrow X.$$

Since  $\Omega\tilde{X}$  and  $\tilde{X}$  are connected, we see that

$$K(n)_*(\Omega X) \simeq K(n)_*(\Omega\tilde{X}) \otimes K(n)_*[\pi_1(X)].$$

From this we see that it is enough to restrict our attention to simply connected spaces; and they all have Postnikov decompositions.

We recall some basic facts about Postnikov towers. Let  $X$  be a simply connected space. Then one has a diagram of the form

$$\begin{array}{ccccccc} X = X_2 & \longleftarrow & X_3 & \longleftarrow & X_4 & \longleftarrow & \dots \\ \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \\ K_2 & & K_3 & & K_4 & & \end{array}$$

where  $K_s = K(\pi_s(X), s)$ ,  $f_s$  is a map inducing an isomorphism in the bottom homotopy group, and  $X_{s+1}$  is the fibre of  $f_s$ .  $X_{s+1}$  is the  $s$ -connected cover of  $X$ . Theorem 1.1 is a slightly weaker  $p$ -local form of the following.

**Theorem 2.1** *Suppose  $X$  has only finitely many nontrivial homotopy groups. Assume that  $\pi_q(X)$ ,  $q > 1$ , has finitely generated torsion free quotient, and finitely generated subgroup of exponent  $p^k$  for each  $p$  and  $k$ . Then for each  $s \geq 1$  there is a natural Hopf algebra extension*

$$K(n)_* \longrightarrow K(n)_*(\Omega X_{s+1}) \longrightarrow K(n)_*(\Omega X_s) \longrightarrow K(n)_*(\Omega K_s) \longrightarrow K(n)_*.$$

*In particular the map  $K(n)_*(\Omega X_s) \rightarrow K(n)_*(\Omega X)$  is always one-to-one.*

*If  $X$  is an  $H$ -space, then we have a natural isomorphism of Hopf algebras:*

$$K(n)_*(\Omega X) \simeq \bigotimes_{i \geq 2} K(n)_*(K(\pi_i(X), i - 1)).$$

*If  $X$  is not an  $H$ -space, the above isomorphism is still valid additively.*

Although this result implies Theorem 1.1 in the Introduction, it is slightly more general. In particular, it allows for homotopy groups with summands like  $\mathbf{Q}/\mathbf{Z}_{(p)}$  which we need. The definition of the filtration in Theorem 1.1 comes from the  $s - 1$ -connected cover,  $X_s$ :

$$F_{-s} \equiv \text{im } K(n)_*(\Omega X_s) \longrightarrow K(n)_*(\Omega X).$$

This filtration is natural. We are not really using the Postnikov construction of a space, which is not natural, but a Postnikov decomposition which is. The Postnikov construction starts with a point and builds up the space one homotopy group at a time. We are starting with the space and taking it apart one homotopy group at a time. Naturality is clear for the maps on the Eilenberg–Mac Lane spaces as they are determined by the maps on the homotopy groups. We want a unique map from  $X_{s+1}$  to  $Y_{s+1}$  if we inductively have a unique map from  $X_s$  to  $Y_s$ . The obstruction to uniqueness is in

$$[X_{s+1}, K(\pi_s(Y), s - 1)]$$

but because  $X_{s+1}$  is  $s$ -connected, this cohomology group is trivial.

**Remark 2.2** *A number of people have asked us questions about how these results can be extended. For example, given a fibration of finite Postnikov systems where the maps all can be delooped several times and the homotopy groups of the fibre map split-injective to the homotopy groups of the total space, what can we say about the Morava  $K$ -theory of everything? Since we know the Morava  $K$ -theory of all the spaces it seems to us that the results and techniques used here should answer any question about this situation.*

In the arguments that follow, it will be convenient to assume that  $\pi_2(X)$  is torsion. In particular, the results of [RW80] imply that  $K(n)_*(\Omega X)$  is concentrated in even dimensions in this case. This assumption is harmless for the following reason. In general (subject to the hypotheses of Theorem 2.1) we have a fibre sequence

$$X' \longrightarrow X \xrightarrow{f} L_2$$

where  $L_2 = K(\pi_2(X)/\text{Torsion}, 2)$  and  $\pi_2(f)$  is surjective. Then  $\pi_2(X')$  is the torsion subgroup of  $\pi_2(X)$ ,  $\Omega L_2$  is a finite product of circles, and

$$\Omega X \simeq \Omega X' \times \Omega L_2 \tag{2.3}$$

because we can lift the maps of the circles to homotopy generators using the  $H$ -space structure. Hence it suffices to compute  $K(n)_*(\Omega X')$ .

### The rational case

The corresponding result for rational homology,  $K(0)$ , is classical and we will sketch its proof now, since it is a model for the proof of Theorem 2.1. Since  $X$  has only finitely many nontrivial homotopy groups,  $X_s$  is contractible for large  $s$  and we can argue by downward induction on  $s$ . Suppose  $\Omega X_{s+1}$  has the prescribed rational homology, and consider the fibre sequence

$$\Omega^2 K_s \xrightarrow{j_s} \Omega X_{s+1} \longrightarrow \Omega X_s. \tag{2.4}$$

For our inductive step we need to prove that

$$H_*(\Omega X_s) \simeq H_*(\Omega X_{s+1}) \otimes H_*(\Omega K_s), \quad (2.5)$$

where all homology groups have rational coefficients. When  $X$  is an  $H$ -space, we need the above isomorphism to be one of Hopf algebras. Otherwise it is an isomorphism of coalgebras.

For our calculation we need the bar spectral sequence for a principle fibration:

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array} \quad (2.6)$$

which has

$$E_{*,*}^2 \simeq \mathrm{Tor}_{*,*}^{K(n)_*(F)}(K(n)_*(E), K(n)_*) \Rightarrow K(n)_*(B). \quad (2.7)$$

If the fibration we use is the loops on a principle fibration, then the bar spectral sequence is a spectral sequence of Hopf algebras. Unfortunately, despite a fascination with the bar construction, e.g. [May72], [May75], [Mey84] and [Mey86], this fact is not in the literature. It can, of course, be patched up easily from what is there about the standard bar construction. Let our fibration be

$$\begin{array}{ccc} \Omega F & \longrightarrow & \Omega E \\ & & \downarrow \\ & & \Omega B \end{array}$$

where (2.6) is a principle fibration. Then  $\Omega F$  has two products, one from  $F$  and one from the loops. When  $\Omega B$  is constructed using the bar construction,

one product can be used in the construction and the other can be used to get a product on the bar construction giving us this spectral sequence as Hopf algebras (the coalgebra structure is no problem). This works for any homology theory with a Künneth isomorphism

In particular, one has the bar spectral sequence converging to  $H_*(\Omega X_s)$  with

$$E^2 = \mathrm{Tor}^{H_*(\Omega^2 K_s)}(H_*(\Omega X_{s+1}), \mathbf{Q}). \quad (2.8)$$

Here the  $H_*(\Omega^2 K_s)$ -module structure on  $H_*(\Omega X_{s+1})$  is induced by the map  $j_s$  of (2.4). We have

**Lemma 2.9** *The map  $j_s$  in (2.4) induces the trivial homomorphism in rational homology.*

*Proof.* The map  $j_s$  is an  $H$ -map, so it must respect the Pontrjagin ring structure in homology. We know that  $H_*(\Omega^2 K_s)$  is generated by elements in dimension  $s - 2$ , while  $\Omega X_{s+1}$  is  $(s - 1)$ -connected, so  $H_*(j_s)$  is trivial.  $\square$

It follows that (2.8) can be rewritten as

$$\begin{aligned} E^2 &= \mathrm{Tor}^{H_*(\Omega^2 K_s)}(H_*(\Omega X_{s+1}), \mathbf{Q}) \\ &= \mathrm{Tor}^{H_*(\Omega^2 K_s)}(\mathbf{Q}, \mathbf{Q}) \otimes H_*(\Omega X_{s+1}) \\ &= H_*(\Omega K_s) \otimes H_*(\Omega X_{s+1}) \end{aligned} \quad (2.10)$$

The rational homology bar spectral sequence collapses for Eilenberg–Mac Lane spaces and has no extension problems, which explains the last step above. Now it follows formally that the spectral sequence collapses, because differentials must lower filtration by at least 2, but  $H_*(\Omega X_{s+1})$  is concentrated in filtration 0, and  $H_*(\Omega K_s)$  is generated by elements in filtration 1. Thus one gets the desired extension of Hopf algebras

$$\mathbf{Q} \longrightarrow H_*(\Omega X_{s+1}) \longrightarrow H_*(\Omega X_s) \longrightarrow H_*(\Omega K_s) \longrightarrow \mathbf{Q}. \quad (2.11)$$

If  $X$  is an H-space, then so is  $X_s$ , so  $H_*(\Omega X_s)$  is bicommutative. The structure of graded connected bicommutative Hopf algebras over  $\mathbf{Q}$  is well known (see [MM65, Section 7]). In particular, we have the following splitting theorem.

**Theorem 2.12** *Let  $A$  be a graded connected bicommutative Hopf algebra over  $\mathbf{Q}$ . Then there is a canonical Hopf algebra isomorphism*

$$A \simeq \bigotimes_{k>0} A_k$$

where  $A_k$  is generated by primitive elements in dimension  $k$ . Moreover,  $A_k$  is a polynomial algebra for  $k$  even and an exterior algebra for  $k$  odd.

It follows that the extension (2.11) is split when  $X$  is an H-space and the rational case of Theorem 2.1 is proved.

### The Morava K-theory case for torsion spaces

Now we will give the proof of Theorem 2.1 under the additional assumption that  $\pi_*(X)$  is all torsion. The general setup is the same as in the rational case. The Morava  $K$ -theory of Eilenberg–Mac Lane spaces was computed in [RW80]. We have the bar spectral sequence as in the rational case, and we have the following analog of Lemma 2.9.

**Lemma 2.13** *The map  $j_s$  in (2.4) induces the trivial homomorphism in Morava  $K$ -theory. (Here we do not require that  $\pi_*(X)$  be all torsion.)*

*Proof.* (See the introduction for  $\overline{K(n)}_*(-)$ .) We will make use of Theorems 1.11 and 1.14. We are studying the map

$$\overline{K(n)}_*(\Omega^2 K_s) \xrightarrow{\overline{K(n)}_*(j_s)} \overline{K(n)}_*(\Omega X_{s+1}).$$

Recall that we are using downward induction on  $s$  so we can assume Theorem 2.1 for all  $t > s$ . Assume our map is nontrivial. Choose the largest  $t$  so that its image is contained in  $K(n)_*(\Omega X_t)$ . Then the composition

$$\overline{K(n)}_*(\Omega^2 K_s) \longrightarrow \overline{K(n)}_*(\Omega X_t) \longrightarrow \overline{K(n)}_*(\Omega X_t) // \overline{K(n)}_*(\Omega X_{t+1}) = \overline{K(n)}_*(\Omega K_t)$$

must be nontrivial. This is a Hopf algebra map, and both the source and target are subject to the splitting theorem 1.11, with the constraints imposed by 1.14. The factors of  $\overline{K(n)}_*(\Omega^2 K_s)$  correspond to orbits  $\gamma$  with  $\bar{\alpha}_p(\gamma) = s - 2$  or  $s - 3$ , while those of  $\overline{K(n)}_*(\Omega K_t)$  have  $\bar{\alpha}_p(\gamma) = t - 1$  or  $t - 2$ . These orbits are distinct since  $s < t \leq n + 1$ . Since there are no nontrivial Hopf algebra homomorphisms between factors corresponding to distinct orbits, the result follows.  $\square$

It follows that the analog of (2.10) holds, namely, in the bar spectral sequence,

$$E^2 = \mathrm{Tor}^{K(n)_*(\Omega^2 K_s)}(K(n)_*, K(n)_*) \otimes K(n)_*(\Omega X_{s+1}).$$

Now,  $\mathrm{Tor}^{K(n)_*(\Omega^2 K_s)}(K(n)_*, K(n)_*)$  is the  $E^2$ -term of the bar spectral sequence converging to  $K(n)_*(\Omega K_s)$ , and this was completely determined in [RW80]. There is a map to this spectral sequence from the one we are studying, given by the following commutative diagram, in which each row is a fibre sequence.

$$\begin{array}{ccccc} \Omega^2 K_s & \xrightarrow{j_s} & \Omega X_{s+1} & \longrightarrow & \Omega X_s \\ \downarrow = & & \downarrow & & \downarrow \\ \Omega^2 K_s & \longrightarrow & \mathrm{pt.} & \longrightarrow & \Omega K_s \end{array}$$

In our spectral sequence (the one for the top row) the extra factor of  $K(n)_*(\Omega X_{s+1})$  is concentrated in the even degrees (by induction) of filtration 0. In a spectral sequence of Hopf algebras the basic differentials must go from generators to primitives, see [Smi70, page 78]. In the bar spectral sequence, differentials must start in filtration greater than one. All generators here are in even degrees and so must go to odd degree primitives, all of which are in filtration one [RW80, Theorem 11.5]. So, the start and finish of the generating differentials are all in the part which maps isomorphically to the bottom row. It follows that in our spectral sequence,

$$E^\infty = E^0 K(n)_*(\Omega K_s) \otimes K(n)_*(\Omega X_{s+1}),$$

where  $E^0 K(n)_*(\Omega K_s)$  denotes the  $E^\infty$ -term of the lower spectral sequence, which was determined in [RW80]. In particular, our  $E^\infty$ -term is concentrated in even dimensions.

Again the edge homomorphism gives us the desired Hopf algebra extension, namely

$$K(n)_* \longrightarrow K(n)_*(\Omega X_{s+1}) \longrightarrow K(n)_*(\Omega X_s) \longrightarrow K(n)_*(\Omega K_s) \longrightarrow K(n)_*. \quad (2.14)$$

The following result is where we need the assumption that  $\pi_*(X)$  is torsion.

**Lemma 2.15** *When  $X$  is an  $H$ -space with torsion homotopy, the extension (2.14) is split naturally.*

*Proof.* We use Theorems 1.11 and 1.14, and assume inductively that  $\overline{K(n)}_*(\Omega X_{s+1})$  is as advertised. This means that its factors under Theorem 1.11 all correspond to orbits  $\gamma$  with  $\bar{\alpha}_p(\gamma) \geq s$ , while the orbits of  $\overline{K(n)}_*(\Omega K_s)$  have  $\bar{\alpha}_p(\gamma) = s - 1$ . Thus Theorem 1.13 assures us that the extension is split naturally.  $\square$

Notice that the proof above would fail if the torsion subgroup of  $\pi_s(X)$  and the torsion free quotient of  $\pi_{s+1}(X)$  were both nontrivial, because in that

case, both  $\overline{K(n)}_*(\Omega X_{s+1})$  and  $\overline{K(n)}_*(\Omega K_s)$  could have factors corresponding to the same orbit  $\gamma$ . In particular, there is a short exact sequence of Hopf algebras,

$$\mathbf{F}_p \rightarrow \overline{K(n)}_*(K(\mathbf{Z}/(p^i), j)) \rightarrow \overline{K(n)}_*(K(\mathbf{Z}_{(p)}, j+1)) \xrightarrow{p^i} \overline{K(n)}_*(K(\mathbf{Z}_{(p)}, j+1)) \rightarrow \mathbf{F}_p,$$

which shows how maps can drop filtration (from  $-j$  to  $-(j+1)$ ). This example prevents us from having a natural Hopf algebra splitting in general.

We do have an unnatural splitting for the general case however, and we produce that now.

### Removing the torsion condition

We have given the proof of Theorem 2.1 in the case when  $\pi_*(X)$  is all torsion. We needed the torsion condition to get the desired Hopf algebra structure when  $X$  is an H-space. We did *not* need it to show that  $K(n)_*(\Omega X_s)$  is a subalgebra of  $K(n)_*(\Omega X)$ . Now we will describe an alternate approach to the Hopf algebra question which does not require  $\pi_*(X)$  to be torsion.

We need to use the *rationalization*  $X\mathbf{Q}$  of  $X$ . For an  $H$ -space (more generally, a nilpotent space),  $Y$ , one has a fibre sequence

$$TY \longrightarrow Y \longrightarrow Y\mathbf{Q}$$

where  $\pi_*(Y\mathbf{Q}) = \pi_*(Y) \otimes \mathbf{Q}$  and  $\pi_*(TY)$  is all torsion. For a  $p$ -local H-space  $Y$  (such as  $\Omega X_{(p)}$ ),  $Y\mathbf{Q}$  is the homotopy direct limit of

$$Y \xrightarrow{[p]} Y \xrightarrow{[p]} Y \xrightarrow{[p]} \dots$$

where  $[p]$  is the H-space  $p^{\text{th}}$  power map. In this case  $Y\mathbf{Q}$  and  $TY$  are both H-spaces.

Now for  $X$  as in Theorem 2.1, we have a fibre sequence

$$\Omega^2 X\mathbf{Q} \longrightarrow \Omega TX \xrightarrow{i} \Omega X. \tag{2.16}$$

Before we can proceed, we need to bring in the generalized Atiyah–Hirzebruch spectral sequence, see [Dol62] and [Dye69]. For a fibration like (2.6), we have the Atiyah–Hirzebruch–Serre spectral sequence:

$$E^2 \simeq H_*(B; K(n)_*(F)) \implies K(n)_*(E). \quad (2.17)$$

The map of  $E$  to  $B$  maps this spectral sequence to the usual Atiyah–Hirzebruch spectral sequence:

$$E^2 \simeq H_*(B; K(n)_*) \implies K(n)_*(B). \quad (2.18)$$

**Lemma 2.19** *If  $\pi_2(X) \otimes \mathbf{Q} = 0$ , then the map  $i$  of (2.16) induces an isomorphism in  $K(n)_*$  for all  $n > 0$ .*

As remarked above (2.3), this assumption on  $\pi_2(X)$  can be made without loss of generality.

*Proof of Lemma 2.19.* We will use the Atiyah–Hirzebruch spectral sequence for Morava K-theory to compute  $K(n)_*(\Omega TX)$ . We have the following commutative diagram in which each row is a fibre sequence.

$$\begin{array}{ccccc}
 \Omega^2 X \mathbf{Q} & \longrightarrow & \Omega TX & \xrightarrow{i} & \Omega X \\
 \downarrow & & \downarrow i & & \downarrow = \\
 \text{pt.} & \longrightarrow & \Omega X & \xrightarrow{=} & \Omega X
 \end{array} \quad (2.20)$$

The hypothesis on  $\pi_2(X)$  implies that  $\Omega^2 X \mathbf{Q}$  is path connected. We know that any rational loop space is a product of rational Eilenberg–Mac Lane spaces. From [RW80, Corollary 12.2] we have

$$\operatorname{dir} \lim K(n)_*(K(\mathbf{Z}/(p^i), q)) \simeq K(n)_*(K(\mathbf{Z}, q + 1)). \quad (2.21)$$

We see that iterating  $[p]$  on the left kills everything, so the product of rational Eilenberg–Mac Lane spaces has the Morava K-theory of a point. This means that the left vertical arrow in (2.20) is a  $K(n)_*$ -equivalence. The right vertical map is a homology equivalence (since it is the identity map) so we get an isomorphism of Atiyah–Hirzebruch–Serre spectral sequences, so  $i$  is a  $K(n)_*$ -equivalence.  $\square$

Now the torsion case of Theorem 2.1 tells us that

$$K(n)_*(\Omega TX) \simeq \bigotimes_{i \geq 2} K(n)_*(K(\pi_i(TX), i - 1)),$$

and this is  $K(n)_*(\Omega X)$  by Lemma 2.19. Thus Theorem 2.1 will follow from

**Lemma 2.22** *For a simply connected space  $X$  with  $\pi_2(X)$  torsion,*

$$\bigotimes_{i \geq 2} K(n)_*(K(\pi_i(X), i - 1)) \simeq \bigotimes_{i \geq 2} K(n)_*(K(\pi_i(TX), i - 1)).$$

*as Hopf algebras.*

*Proof.* There is a split short exact sequence

$$0 \longrightarrow \pi_{i+1}(X) \otimes \mathbf{Q}/\mathbf{Z} \longrightarrow \pi_i(TX) \longrightarrow \operatorname{Tor}_1(\pi_i(X), \mathbf{Q}/\mathbf{Z}) \longrightarrow 0.$$

Note that  $\operatorname{Tor}_1(\pi_i(X), \mathbf{Q}/\mathbf{Z})$  is the torsion subgroup of  $\pi_i(X)$ , while  $\pi_{i+1}(X) \otimes \mathbf{Q}/\mathbf{Z}$  is the tensor product of  $\mathbf{Q}/\mathbf{Z}$  with the torsion free quotient of  $\pi_{i+1}(X)$ . With this in mind, we also have a split short exact sequence

$$0 \longrightarrow \operatorname{Tor}_1(\pi_i(X), \mathbf{Q}/\mathbf{Z}) \longrightarrow \pi_i(X) \longrightarrow \pi_i(X)/\operatorname{Torsion} \longrightarrow 0.$$

According to (2.21)

$$K(n)_*(K(\mathbf{Z}, i + 1)) \simeq K(n)_*(K(\mathbf{Q}/\mathbf{Z}, i)) \quad \text{for } i \geq 1,$$

so we have

$$K(n)_*(K(\pi_{i+1}(X) \otimes \mathbf{Q}/\mathbf{Z}, i - 1)) \simeq K(n)_*(K(\pi_{i+1}(X)/\text{Torsion}, i)).$$

It follows that

$$\begin{aligned} & \bigotimes_{i \geq 2} K(n)_*(K(\pi_i(TX), i - 1)) \\ &= \bigotimes_{i \geq 2} K(n)_*(K(\pi_{i+1}(X) \otimes \mathbf{Q}/\mathbf{Z}, i - 1)) \\ & \quad \oplus \\ & \bigotimes_{i \geq 2} K(n)_*(K(\text{Tor}_1(\pi_i(X), \mathbf{Q}/\mathbf{Z}), i - 1)) \\ &= \bigotimes_{i \geq 2} K(n)_*(K(\pi_{i+1}(X)/\text{Torsion}, i)) \\ & \quad \oplus \\ & \bigotimes_{i \geq 2} K(n)_*(K(\text{Tor}_1(\pi_i(X), \mathbf{Q}/\mathbf{Z}), i - 1)) \\ &= \bigotimes_{i \geq 2} K(n)_*(K(\pi_i(X)/\text{Torsion}, i - 1)) \\ & \quad \oplus \\ & \bigotimes_{i \geq 2} K(n)_*(K(\text{Tor}_1(\pi_i(X), \mathbf{Q}/\mathbf{Z}), i - 1)) \\ &= \bigotimes_{i \geq 2} K(n)_*(K(\pi_i(X), i - 1)). \end{aligned}$$

□

**Remark 2.23** *The above process lost us our naturality in the splitting, but not if all of the homotopy groups are free or torsion. The problems only come up if we try to mix them.*

*Proof of Theorem 1.4.* First let us assume that  $\pi_{n+1}(X) = \pi_{n+2}(X) = 0$ . Theorem 1.1 tells us that  $K(n)_*(\Omega X)$  is trivial and our result follows from the bar spectral sequence.

**Remark 2.24** *We do not have to use Theorem 1.1 here at all. We can do our downward induction on the Postnikov system directly. Everything will be trivial so there is no difficulty showing the maps are also trivial and the spectral sequence is trivial. One can just use Lemma 3.3 over and over again. We use this result several times in our study of spectra; the point is that those results are independent of Theorem 1.1.*

Next we consider the case where  $\pi_{n+1}(X) = 0$ . If  $\pi_{n+2}(X) = 0$  or is torsion, the same proof works. It is only if  $\pi_{n+2}(X)$  has copies of  $\mathbf{Z}_{(p)}$  in it that we could have a problem. We take our usual fibration with map:

$$\begin{array}{ccccc}
 \Omega K_{n+2} & \longrightarrow & X_{n+3} & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow \\
 = & & & & \\
 \Omega K_{n+2} & \longrightarrow & \text{pt.} & \longrightarrow & K_{n+2}.
 \end{array}$$

Since  $\pi_{n+1}(X_{n+3}) = \pi_{n+2}(X_{n+3}) = 0$ , we have  $K(n)_*(X_{n+3})$  is trivial. Thus we get an isomorphism on the  $E^2$  terms of the two bar spectral sequences. Thus the  $E^\infty$  terms must be isomorphic as well. However, since we know, from [RW80, Theorem 12.3], that

$$\text{Tor}^{K(n)_*(K(\mathbf{Z}_{(p)}, n+1))}(K(n)_*, K(n)_*) = K(n)_*,$$

we get our result.

All we have left to deal with is the case where  $\pi_{n+1}(X)$  is finite. The argument is exactly the same as that just given except that the Tor is trivial by [RW80, Theorem 11.5].  $\square$

### 3 Strongly $K(n)_*$ -acyclic connective spectra

Throughout this section  $n$  will be assumed to be positive. For a connective spectrum  $X$ ,  $\underline{X}_m$  for  $m \geq 0$  will denote the  $m^{\text{th}}$  space in the associated  $\Omega$ -spectrum. Be alert to our (unusual) convention that  $\underline{X}_m$  have its bottom cell in dimension  $m$ .

**Definition 3.1** *A connective spectrum  $X$  is **strongly  $K(n)_*$ -acyclic** if  $\underline{X}_m$  is  $K(n)_*$ -acyclic for some  $m \in \mathbf{Z}$ .*

One might guess that any  $K(n)_*$ -acyclic spectrum is strongly  $K(n)_*$ -acyclic, but we will see below, (3.9), that this is not the case. We will show, Theorem 3.7, that for such an  $X$  each space  $\underline{X}_m$  is  $K(n)_*$ -equivalent to a suitable finite Postnikov system.

**Remark 3.2** *We take a moment to show that we need not restrict ourselves to positive spaces in the  $\Omega$ -spectrum, such as in Theorem 1.6. If  $m \geq 0$ , then we see from the bar spectral sequence that  $\underline{X}_m$   $K(n)_*$ -acyclic implies  $\underline{X}_{m+1}$  is  $K(n)_*$ -acyclic. If we have a negative number, write  $\underline{X}_{-m}$ . Let  $X\langle m+1 \rangle$  be the stable  $m$ -connected cover of  $X$ . Then we have a stable cofibration:*

$$\Sigma^m X\langle m+1 \rangle \longrightarrow X \longrightarrow F$$

where  $F$  is a finite Postnikov system and we have  $X\langle m+1 \rangle_0 = \underline{X}_{-m}$ . If this is  $K(n)_*$ -acyclic, then  $\underline{X}\langle m+1 \rangle_k$  is  $K(n)_*$ -acyclic for all  $k \geq 0$ . We have an unstable fibration

$$\underline{F}_{k-1} \longrightarrow \underline{X}\langle m+1 \rangle_{m+k} \longrightarrow \underline{X}_k.$$

For big  $k$ ,  $\underline{F}_{k-1}$  is also  $K(n)_*$ -acyclic by Theorem 1.4, so  $\underline{X}_k$  is  $K(n)_*$ -acyclic by the next lemma.

**Lemma 3.3** *Let*

$$F \xrightarrow{i} E \xrightarrow{j} B$$

*be a fibration with  $F$   $K(n)_*$ -acyclic. Then the map  $j$  is a  $K(n)_*$ -equivalence.*

*Proof.* There is an Atiyah–Hirzebruch–Serre spectral sequence, (2.17), converging to  $K(n)_*(E)$  with

$$E_2 = H_*(B; K(n)_*(F)).$$

It maps to the usual Atiyah–Hirzebruch spectral sequence converging to  $K(n)_*(B)$ . Since  $F$  is  $K(n)_*$ -acyclic, this map is an isomorphism, giving the desired result.  $\square$

**Proposition 3.4** *Let  $m \geq 0$ ,  $X$  be a connective spectrum with  $\underline{X}_m$   $K(n)_*$ -acyclic, and let  $Y$  be any connective spectrum. Then  $(\underline{X \wedge Y})_m$  is also  $K(n)_*$ -acyclic.*

Recall here that our convention is that  $\underline{X}_m$  and  $(\underline{X \wedge Y})_m$  have the same connectivity.

*Proof.* We will argue by skeletal induction on  $Y$ . Assume for simplicity that the bottom cell of  $Y$  is in dimension 0, so the 0-skeleton  $Y^0$  is a wedge of spheres. Thus  $(\underline{X \wedge Y^0})_m$  is a product of  $\underline{X}_m$ s and is therefore  $K(n)_*$ -acyclic. For  $k > 0$  consider the cofibre sequence

$$\Sigma^{-1} X \wedge Y^k / Y^{k-1} \longrightarrow X \wedge Y^{k-1} \longrightarrow X \wedge Y^k,$$

which we abbreviate by  $A \rightarrow B \rightarrow C$ . Now  $\underline{A}_{m+k-1}$  (which may be contractible) is  $K(n)_*$ -acyclic by a similar argument, and we can assume inductively that  $\underline{B}_m$  is  $K(n)_*$ -acyclic. We have a fibration

$$\underline{A}_{m+k-1} \xrightarrow{i} \underline{B}_m \xrightarrow{j} \underline{C}_m$$

and  $\underline{C}_m$  is  $K(n)_*$ -acyclic by 3.3. This is true for all  $k > 0$ , so  $(\underline{X \wedge Y})_m$  is  $K(n)_*$ -acyclic as claimed because it is the direct limit of  $K(n)_*$ -acyclic spaces.  $\square$

We need a standard result which we include here for completeness.

**Lemma 3.5** *Let  $X$  be a connective spectrum and  $H$  be the integral Eilenberg–Mac Lane spectrum. Then  $X \wedge H$  is a product of Eilenberg–Mac Lane spectra.*

*Proof.* We argue by skeletal induction. We have

$$\Sigma^{-1}X^k/X^{k-1} \wedge H \longrightarrow X^{k-1} \wedge H \longrightarrow X^k \wedge H.$$

By induction we have  $X^{k-1} \wedge H$  is the product of Eilenberg–Mac Lane spectra with homotopy  $H_*(X^{k-1}; \mathbf{Z})$ . The first term is just a bunch of  $\Sigma^{k-1}H$ s and so factors through  $K(H_{k-1}(X^{k-1}), k-1)$  in the second term so the third term is as claimed.  $\square$

Now we can get some control over the connectivity  $m$ .

**Lemma 3.6** *Let  $X$  be a connective strongly  $K(n)_*$ -acyclic spectrum with bottom cell in dimension 0. Then the space  $\underline{X}_{n+3}$  is  $K(n)_*$ -acyclic.*

*Proof.* We assume that all spectra and spaces in sight are localized at the prime  $p$ . Let  $H$  denote the integer Eilenberg–Mac Lane spectrum, and let  $\overline{H}$  denote the fibre of the map  $S^0 \rightarrow H$ .

$$\overline{H} \longrightarrow S^0 \longrightarrow H$$

The bottom cell of  $\overline{H}$  is in dimension  $q-1$ , where  $q = 2(p-1)$ . Thus the cofibre sequence

$$\Sigma^{-1}X \wedge H \longrightarrow X \wedge \overline{H} \longrightarrow X$$

gives a fibre sequence

$$\underline{(X \wedge H)}_{m-1} \xrightarrow{i} \underline{(X \wedge \overline{H})}_{m+q-1} \xrightarrow{j} \underline{X}_m.$$

(Recall our convention!) Now  $\underline{(X \wedge H)}_{m-1}$  is an  $(m-2)$ -connected product of Eilenberg–Mac Lane spaces, so it is  $K(n)_*$ -acyclic for  $m-1 > n+1$ . Thus  $j$  is a  $K(n)_*$ -equivalence for  $m \geq n+3$ . Iterating this argument we see that

$\underline{X}_{n+3}$  is  $K(n)_*$ -equivalent to  $(X \wedge \overline{H}^{(s)})_{n+3+s(q-1)}$  for each  $s > 0$ . The latter is  $K(n)_*$ -acyclic for some  $s$  by Proposition 3.4, so  $\underline{X}_{n+3}$  is also  $K(n)_*$ -acyclic.  $\square$

We can use this lemma to prove the following.

**Theorem 3.7** *Let  $X$  be a connective strongly  $K(n)_*$ -acyclic spectrum with bottom cell in dimension 0. Let  $X \rightarrow F$  be a map to a finite Postnikov system which is an equivalence through dimension  $n+1$ . Then  $\underline{X}_m$  is  $K(n)_*$ -equivalent to  $\underline{F}_m$  for all  $m \geq 0$ . In particular,  $\underline{X}_{n+2}$  is  $K(n)_*$ -acyclic, and if  $\pi_0(X)$  is torsion,  $\underline{X}_{n+1}$  is  $K(n)_*$ -acyclic.*

*Proof.* Stably we have a fibre sequence

$$X' \longrightarrow X \longrightarrow F$$

where  $X'$  is a connected cover of  $X$  having bottom cell above dimension  $n+1$ . Consider the fibre sequence

$$\underline{F}_{m-1} \longrightarrow \underline{X}'_{m'} \longrightarrow \underline{X}_m$$

(where  $m' > m+n+1$  depends on the connectivity of  $X'$ ). From Theorem 1.4 we know that  $F$  is strongly  $K(n)_*$ -acyclic. Since  $X$  is also strongly  $K(n)_*$ -acyclic we can use Lemma 3.3 to see that  $\underline{X}'_{m'}$  is also strongly  $K(n)_*$ -acyclic. Hence  $\underline{X}'_{n+3}$  is  $K(n)_*$ -acyclic by Lemma 3.6.

Now consider the fibre sequence

$$\underline{X}'_{m'} \xrightarrow{i} \underline{X}_m \xrightarrow{j} \underline{F}_m.$$

For positive  $m$ ,  $m' > m+n+1 \geq n+2$ , so  $\underline{X}'_{m'}$  is  $K(n)_*$ -acyclic. By Lemma 3.3,  $j$  is a  $K(n)_*$ -equivalence as claimed. In particular,  $\underline{X}_{n+2}$  is  $K(n)_*$  equivalent to  $\underline{F}_{n+2}$ , which by Theorem 1.4 is  $K(n)_*$ -acyclic. Since  $X$  was arbitrary strongly  $K(n)_*$ -acyclic, we also know that  $\underline{X}'_{n+2}$  is  $K(n)_*$ -acyclic, so  $j$  is also a  $K(n)_*$ -equivalence for  $m = 0$ .  $\square$

Pete Bousfield has generalized this in more than one way. His results are not restricted to Morava  $K$ -theories and he does not need to work with  $\Omega$ -spectra. In a short note to us he derived these results from [Bou94]. More recently, these have been made explicit in [Bou96b, Section 7]. Restricting our attention to Morava  $K$ -theories, his result of interest to us is:

**Theorem 3.8 (Bousfield)** *An  $(n+2)$ -connected  $H$ -space  $Y$  is  $K(n)_*$ -acyclic if and only if  $\Omega Y$  is  $K(n)_*$ -acyclic.*

The following example shows that not all connective  $K(n)_*$ -acyclic spectra are strongly  $K(n)_*$ -acyclic.

**Example 3.9** *Let  $X$  be the spectrum  $BP\langle n-1 \rangle$  from [Wil75] and [JW73] for  $n > 1$ . The third author showed in [Wil75] that  $\underline{X}_m$  has torsion free homology and therefore is not  $K(n)_*$ -acyclic for  $m = 2(p^n - 1)/(p - 1)$ . This would contradict Lemma 3.6 if  $X$  were strongly  $K(n)_*$ -acyclic, so it is not. It is, however,  $K(n)_*$ -acyclic. This can be seen most easily by considering the exterior Hopf algebra on the Milnor Bockstein,  $Q_n$ , as a sub-Hopf algebra of the cohomology*

$$H^*(BP\langle n-1 \rangle; \mathbf{Z}/(p)) \simeq \mathcal{A}/\mathcal{A}(Q_0, Q_1, \dots, Q_{n-1}).$$

*Then, by Milnor–Moore, 4.4, [MM65],  $H^*(BP\langle n-1 \rangle; \mathbf{Z}/(p))$  is free over  $E(Q_n)$  and the first differential in the Atiyah–Hirzebruch spectral sequence for  $K(n)_*(BP\langle n-1 \rangle)$  kills everything and we find  $BP\langle n-1 \rangle$  is  $K(n)_*$ -acyclic.*

**Theorem 3.10**

- (i) *Let  $X \rightarrow Y \rightarrow Z$  be a cofibre sequence of connective spectra. If any two of  $X$ ,  $Y$  and  $Z$  are strongly  $K(n)_*$ -acyclic, then so is the third.*
- (ii) *Any retract  $X$  of a connective strongly  $K(n)_*$ -acyclic spectrum  $Y$  is also strongly  $K(n)_*$ -acyclic.*

(iii) Any connective direct limit  $X$  of connective strongly  $K(n)_*$ -acyclic spectra is strongly  $K(n)_*$ -acyclic.

*Proof.* (i) is an easy consequence of Lemma 3.3. For (ii) note that  $\underline{X}_m$  is a retract of  $\underline{Y}_{m'}$  for suitable  $m'$ . For (iii) we can assume that each spectrum in the direct system has bottom cell in dimension  $\geq 0$ , so  $\underline{Y}_{n+2}$  is a direct limit of  $K(n)_*$ -acyclic spaces.  $\square$

We have so far produced no examples of strongly  $K(n)_*$ -acyclic spectra other than finite Postnikov systems.

**Proposition 3.11** *Any finite  $K(n)_*$ -acyclic spectrum  $X$  is strongly  $K(n)_*$ -acyclic.*

*Proof.*  $X$  is a suspension spectrum of a (connected) finite complex  $Y$ , so for some  $m$ ,  $\underline{X}_m = QY$ , where

$$QY = \varinjlim_k \Omega^k \Sigma^k Y.$$

Since  $K(n)_*(Y) \simeq K(n)_*(X)$ ,  $Y$  is  $K(n)_*$ -acyclic. All we have to do is show that  $Y$   $K(n)_*$ -acyclic implies that  $QY$  is  $K(n)_*$ -acyclic. There is a stable splitting, [Sna74], for  $QY$ . Each piece of this stable splitting is identified explicitly as something called  $e[\mathcal{C}(j), \Sigma_j, Y]$  from [May72, Proposition 2.6(ii), page 14]. Since  $\mathcal{C}(j)$  is contractible, [May72, page 5], this is the same as  $D_{\Sigma_j} Y$ ; for which there is a spectral sequence, [CLJ76, Corollary 2.4, page 7]:

$$E^2 \simeq H_*(\Sigma_j; K(n)_*(Y^{(j)})) \Rightarrow K(n)_*(D_{\Sigma_j} Y)$$

where  $Y^{(j)}$  is the  $j^{\text{th}}$  smash product. For reduced Morava  $K$ -theory,  $K(n)_*(Y^{(j)}) = 0$  because we have a Künneth isomorphism. Thus each piece of the stable splitting is  $K(n)_*$ -acyclic and we have that  $QY$  is  $K(n)_*$ -acyclic. This is well known to those familiar with this but, to a novice, perhaps a bit difficult to dig out of [Sna74].  $\square$

**Corollary 3.12** *Any connective spectrum which is a direct limit of finite  $K(n)_*$ -acyclic spectra is strongly  $K(n)_*$ -acyclic.*

*Proof.* This follows immediately from Theorem 3.10 and Proposition 3.11.  $\square$

Now recall the localization functors  $L_n$  of [Rav84] and  $L_n^f$  of [Rav93]. The former is Bousfield localization with respect to  $E(n)$ , while the latter is constructed in such a way that the fibre of the map  $X \rightarrow L_n^f X$  is a direct limit of finite  $K(n)_*$ -acyclic spectra. Thus we get

**Corollary 3.13** *A connective spectrum  $X$  for which  $L_n^f X$  is contractible is strongly  $K(n)_*$ -acyclic.*

*Proof.* Since  $L_n^f X$  is contractible, the fibre of the map  $X \rightarrow L_n^f X$  is equivalent to  $X$ . Since  $X$  is a connective spectrum which is now the direct limit of finite  $K(n)_*$ -acyclic spectra,  $X$  is strongly  $K(n)_*$ -acyclic by Corollary 3.12.  $\square$

There is a natural transformation  $\lambda_n : L_n^f \rightarrow L_n$ . The telescope conjecture, which is known to be false for  $n = 2$ , see [Rav] and [Rav92b], is equivalent to the assertion that  $\lambda_n$  is an equivalence. It is shown that for  $n = 2$  there is a spectrum  $X$  for which  $L_n X$  is contractible but  $L_n^f X$  is not. It is necessarily a torsion spectrum (its rational homology must vanish) and its connective cover has the same property. However we do not know if such a spectrum is strongly  $K(n)_*$ -acyclic or not.

We also know of no counterexample to the converse of Corollary 3.13, so perhaps that is an equivalence. Alternatively, one can ask if a connective spectrum  $X$  is strongly  $K(n)_*$ -acyclic if and only if  $E(n)_*(X) = 0$ . In [Rav84, Theorem 2.1(d)] it was shown that  $E(n)_*(X) = 0$  if and only if  $K(i)_*(X) = 0$  for  $0 \leq i \leq n$ . It is also known, [JY80], that  $E(n)_*(X) = 0$  if and only if  $v_n^{-1}BP_*(X) = 0$ .

The following is a consequence of Corollary 3.13.

**Theorem 3.14** *If  $X$  is a connective spectrum with  $E(n)_*(X) = 0$ , then  $BP \wedge X$  is strongly  $K(n)_*$ -acyclic. In particular if  $X$  is also a  $BP$ -module spectrum then it is strongly  $K(n)_*$ -acyclic. The same holds with  $BP$  replaced by any connective spectrum  $E$  with Bousfield class dominated by that of  $BP$ .*

*Proof.* By the smash product theorem [Rav92a, 7.5.6],  $E(n)_*(X) = 0$  if and only if  $X \wedge L_n S^0$  is contractible. We also know [Rav93, Theorem 2.7(iii)] that  $L_n S^0$  and  $L_n^f S^0$  are  $BP_*$ -equivalent and therefore  $E_*$ -equivalent. Thus we have

$$\text{pt.} \simeq E \wedge X \wedge L_n S^0 \simeq E \wedge X \wedge L_n^f S^0 \simeq L_n^f(E \wedge X)$$

(the last equivalence is Theorem 2.7(ii) of [Rav93]) and  $E \wedge X$  is strongly  $K(n)_*$ -acyclic by Corollary 3.13. If  $E$  is a ring spectrum and  $X$  is an  $E$ -module spectrum, then it is a retract of  $E \wedge X$ ,

$$X \simeq S^0 \wedge X \longrightarrow E \wedge X \longrightarrow X$$

so it is strongly  $K(n)_*$ -acyclic by Theorem 3.10(ii).  $\square$

**Corollary 3.15** *If  $X$  is a  $BP$ -module spectrum in which each element of  $\pi_*(X)$  is annihilated by some power of the ideal*

$$I_{n+1} = (p, v_1, v_2, \dots, v_n) \subset BP_*,$$

*then  $X$  is strongly  $K(n)_*$ -acyclic.*

*Proof.* The hypothesis implies that  $v_n^{-1}\pi_*(X) = 0$ , so  $v_n^{-1}BP_*(X) = 0$ , which is equivalent to  $E(n)_*(X) = 0$  as noted above.  $\square$

Examples of spectra satisfying these hypotheses include the  $P(k)$  of [JW75] (with  $\pi_*(P(k)) = BP_*/I_k$ ) for  $k > n$  and spectra obtained from  $BP\langle k \rangle$  by killing an ideal containing some power of  $I_{n+1}$ .

## 4 Morava homology Hopf algebras

We want to study a category of Hopf algebras which includes the objects of our interest: the Morava  $K$ -theory (homology) of homotopy commutative, connected  $H$ -spaces. We need this study to solve Hopf algebra extension problems in the bar spectral sequence during the inductive step of the proof of our main theorem. To do this we have a general Hopf algebra splitting theorem which is of interest in its own right. We want to give particular thanks to Hal Sadofsky for help with this section.

Although our Hopf algebras will be bicommutative, and so give an abelian category, we can run into serious problems because of the cyclical grading we use. In particular, we can have an element with the property  $x = x^p$ . (In  $\overline{K(n)}_*(K(\mathbf{Z}/(p), n))$  for example.) This wreaks havoc with all of the algebra portion of Milnor–Moore [MM65]. Such an algebra has no generator (i.e., indecomposable), and the first proposition of Milnor–Moore is false for our situation. The coalgebra portion of Milnor–Moore fares much better. Before we move on to Hopf algebras we want to indicate why our coalgebras are still nice by making some definitions and reproving a basic result which still holds in our setting.

Let  $A$  be a cocommutative, coassociative, coaugmented coalgebra with counit over a ring,  $R$ . At present we are not concerned with gradings so this could be ungraded, graded or cyclically graded. *To avoid unnecessary complications, we assume that  $A$  is flat over  $R$ .* Let  $J(A)$  be the cokernel of the coaugmentation map:

$$0 \rightarrow R \rightarrow A \rightarrow J(A) \rightarrow 0.$$

Using the iterated coproduct we define an increasing filtration,  $F_q A$ , by

$$F_q A = \ker(A \rightarrow J(A)^{\otimes q+1})$$

for all  $q \geq 0$ . Note that  $F_0 A \simeq R$  and  $F_1 A / F_0 A \simeq P(A)$ , the primitives of  $A$ . We call this the *primitive* filtration of  $A$ . Dualizing Milnor–Moore, [MM65,

page 252], we could call this the coaugmentation filtration. (Milnor and Moore reserve the name *primitive filtration* for their filtration on primitively generated Hopf algebras.)

Following [Boa81] we say a filtration,  $F_*A$ , is *exhaustive* (or exhausts  $A$ ) if every element of  $A$  is in some  $F_qA$ . We say  $A$  is a (*connected*) *homology coalgebra* if  $A$  is a cocommutative, coassociative, coaugmented coalgebra over  $R$  with counit, its primitive filtration exhausts  $A$ , and it is the direct limit of finite dimensional  $R$  sub-coalgebras. The exhaustive condition on  $A$  replaces connectivity in the graded case quite nicely. To justify the name we have the following observation:

**Lemma 4.1** *Let  $E_*(-)$  be a multiplicative homology theory and  $X$  a connected CW complex of finite type such that  $E_*(X)$  is flat over  $E_*$ , then  $E_*(X)$  is a homology coalgebra over  $E_*$ .*

*Proof.* Because  $E_*(X)$  is flat over  $E_*$ , the Künneth isomorphism holds for finite products of  $X$ . The diagonal,  $X \rightarrow X \times X$ , induces a cocommutative, coassociative coalgebra over  $E_*$ . The map of a point into  $X$  (because  $X$  is connected) and the map of  $X$  to a point give the coaugmentation and counit respectively. Again, we need connectivity to show the primitive filtration is exhaustive. An element of  $E_*(X)$  which lives on the 0-cell maps trivially to  $J(E_*(X))$ . Since  $E_*(X)$  is the direct limit of  $E_*(X^q)$  where  $X^q$  is the  $q$ -skeleton, any element  $x$  in  $E_*(X)$  comes from  $E_*(X^q)$  for some  $q$  (this shows the coalgebra is the direct limit of finite dimensional  $E_*$  sub-coalgebras). Getting a cellular approximation to the iterated diagonal map,  $X \rightarrow \prod^{q+1} X$  we see that on some coordinate  $X^q$  maps to the 0-cell and thus our element must be in  $F_qE_*(X)$  and the primitive filtration is exhaustive.  $\square$

We can also prove a standard result which, as we see, does not depend on a grading, but just on having an exhaustive primitive filtration. This is well known and is even somewhere in the algebraic topology literature but we cannot remember where we have seen it. In the *other* literature, it could

well be that Lemma 11.0.1 on page 217 of [Swe69] could prove it; but it is a lot easier to reprove it than to be sure of that.

**Lemma 4.2** *Let  $A \rightarrow B$  be a map of coalgebras where  $A$  is a homology coalgebra, then the map injects if and only if the induced map on primitives injects.*

*Proof.* Since  $P(A) \subset A$ , an injection on  $A$  is automatically an injection on  $P(A)$ . In the other direction our proof is by induction on the degree of the primitive filtration of an element. We are given that  $P(A) \simeq F_1 A$  injects to ground our induction. If we have an element,  $x \in F_q A$  but not in  $F_{q-1} A$ , then the coproduct takes  $x$  to  $\sum x' \otimes x''$  in  $J(A) \otimes J(A)$  where each non-zero  $x'$  and  $x''$  must be in a lower filtration and so they inject to  $J(B) \otimes J(B)$ .  $\square$

We say that  $A$  is a (*connected*) *homology Hopf algebra* if it is an associative Hopf algebra with unit such that the coalgebra structure is that of a homology coalgebra. Clearly, if our  $X$  above is an  $H$ -space, then  $E_*(X)$  is a homology Hopf algebra. If the algebra structure is commutative, we say  $A$  is a *commutative homology Hopf algebra*. If  $X$  above is a homotopy commutative  $H$ -space (e.g. any double loop space) then  $E_*(X)$  is a commutative homology Hopf algebra.

Since the exhaustive condition replaces connectivity so nicely, we can define a Hopf algebra conjugation on homology Hopf algebras. This is done inductively on the primitive filtration. Let  $x \rightarrow x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$ , then  $C(x) = -x - x' C(x'')$  inductively. Following [MM65, Proposition 8.8, page 260], we have  $C \circ C = I_A$  because our coalgebra is cocommutative. The existence of  $C$  is essential to show that commutative homology Hopf algebras are an abelian category, see [Gug66]. We are in a slightly generalized situation over the usual connected bicommutative Hopf algebras of finite type so it is worth a short discussion of our case. If  $A$  is a coalgebra and  $B$  is an algebra, then two maps from  $A$  to  $B$  can be combined to get a third, still following

[MM65, Section 8], by

$$A \rightarrow A \otimes A \xrightarrow{f \otimes g} B \otimes B \rightarrow B.$$

To get an abelian category we need  $\text{Hom}(A, B)$  to be an abelian group. In particular, the above map must be a Hopf algebra map. For  $A \rightarrow A \otimes A$  to be a Hopf algebra map,  $A$  must be cocommutative. For  $B \otimes B \rightarrow B$  to be a Hopf algebra map,  $B$  must be commutative. So, we need the bicocommutativity of commutative homology Hopf algebras to get our composition back in our category. The Hopf algebra conjugation above gives us our inverse and bicocommutativity shows  $\text{Hom}(A, B)$  is an abelian group. Following [MM65, Sections 3 and 4], bicocommutativity allows us to define, for all  $f : A \rightarrow B$ , a kernel

$$A \backslash\! \! \! / f \simeq R \square_B A \simeq A \square_B R$$

and a cokernel

$$B // f \simeq R \otimes_A B \simeq B \otimes_A R.$$

Tensor product is the product in our category and given maps  $f : A \rightarrow B$  and  $g : A \rightarrow C$ , we get  $(f, g) : A \rightarrow A \otimes A \xrightarrow{f \otimes g} B \otimes C$ . We have, see [Bou96a, Appendix, especially Theorem 10.12]:

**Theorem 4.3** *The category of commutative homology Hopf algebras is abelian.*

In the graded connected case over a perfect field of characteristic  $p$ , commutative homology Hopf algebras of finite type have been classified nicely in [Sch70]. This applies to the case where  $E$  is standard mod  $p$  homology theory. Unaware of Schoeller's work, the second author had also embarked on such a project. His project was aborted when he learned about Schoeller's paper and the only physical remains of this project are in [Rav75]. His approach was quite different from hers and quite easily extends to the case at hand, so we are now reaping the benefits of his study in this paper.

What concerns us here are what we will call *commutative Morava homology Hopf algebras*. These are just commutative homology Hopf algebras

over  $\mathbf{F}_p$  which are cyclically graded over  $\mathbf{Z}/(2(p^n - 1))$  for some  $n$ . These arise naturally when you take the Morava  $K$ -theory,  $K(n)_*(X)$ , where  $X$  is a connected homotopy commutative  $H$ -space, and set  $v_n$  equal to one as we do throughout the rest of this paper. We do only what we need with these Hopf algebras here, but we hope to return to the problem of classifying them in a future paper. Because of the cyclic grading there is a much richer, more interesting collection of Hopf algebras than in the standard case. The Morava  $K$ -theory of Eilenberg–Mac Lane spaces in [RW80] supplies lots of examples unlike anything seen in the normal graded case.

**Remark 4.4** *At this stage we must point out a problem and its solution for the prime  $p = 2$ .  $K(n)$  is not a homotopy commutative ring spectrum for  $p = 2$ , so  $K(n)_*(X)$  is not necessarily in our category. However, if  $K(n)_*(X)$  is even degree it is. For a discussion of this problem see the Appendix of [JW82] where, following [Wür77], [RW80] is shown to hold for  $p = 2$ .*

Because of the cyclic grading, it is not unusual to find ourselves dealing with infinite dimensional vector spaces; for example, a polynomial algebra with one primitive generator already gets us into that situation. However, we can use finiteness when we need it because our coalgebras are the direct limit of finite dimensional sub-coalgebras. Many Hopf algebra categories are self dual, an extremely nice property. Unfortunately, ours is not self dual. We can define the dual category, however.

**Proposition 4.5** *Let  $A$  be a commutative homology Hopf algebra over a field  $R$ . It is the direct limit of its finite dimensional subcoalgebras  $A_\alpha$ . Let its dual be defined by*

$$A^* = \varprojlim \mathrm{Hom}_R(A_\alpha, R).$$

*Then  $A^*$  is a compact topological bicommutative Hopf algebra under the inverse limit topology. Moreover  $A$  is the continuous linear dual of  $A^*$ .*

*Proof.* The product and coproduct in  $A^*$  are induced respectively by the coproduct and product in  $A$ .  $A^*$  is compact because it is the inverse limit

of finite dimensional vector spaces. The continuous linear dual of an inverse limit is the direct limit of the continuous linear dual, and

$$\mathrm{Hom}_R(\mathrm{Hom}_R(A_\alpha, R), R) = A_\alpha$$

so the continuous linear dual of  $A^*$  is  $A$ .  $\square$

This result enables us to make the following definition.

**Definition 4.6** *Let  $A$  be a commutative homology Hopf algebra over  $\mathbf{F}_p$ . The **Frobenius map**  $F : A \rightarrow A$  is the Hopf algebra homomorphism that sends each element to its  $p^{\mathrm{th}}$  power. The **Verschiebung map**  $V : A \rightarrow A$  is the dual of the Frobenius map on  $A^*$*

We are ready to look closely at the category of commutative Morava homology Hopf algebras and say what we need to say about it. We propose to split up every Hopf algebra into canonical components. Because it is all we need for this paper, we restrict our attention to evenly graded Hopf algebras. Thus our Morava Hopf algebras are really graded over  $G = \mathbf{Z}/(p^n - 1)$ . Denote by  $EC(n)$  the category of evenly graded commutative Morava homology Hopf algebras (for a given  $n$ ). Let  $H$  be the cyclic group of order  $n$ .  $H$  acts on  $G$  via the  $p^{\mathrm{th}}$  power map. Writing  $G$  additively, the map  $H \times G \rightarrow G$  is given by  $(i, j) \rightarrow p^i j$ . Let  $\gamma$  denote an  $H$ -orbit:

$$\gamma = \{j, pj, p^2j, \dots\} \subset G.$$

Recall Theorem 1.11 from the introduction, which uses these definitions.

**Remark 4.7** *We illustrate the result with the case  $p = 2$  and  $n = 3$ . Then  $H$  and  $G$  have orders 3 and 7 respectively and there are three orbits, namely*

$$\begin{aligned} \gamma_1 &= \{1, 2, 4\}, \\ \gamma_2 &= \{3, 6, 5\} \quad \text{and} \\ \gamma_3 &= \{0\}. \end{aligned}$$

We also know, from [RW80], that for  $i = 1, 2$  or  $3$ ,  $K(3)_*(K(T, i))$  has its primitives in dimensions in  $2\gamma_i$  for any torsion abelian group  $T$ .

We discuss this more later, but in general,  $K(n)_*(K(T, i))$  can have factors with generators in more than one orbit. For example when  $p = 2$ ,  $n = 4$  and  $i = 2$  there are two such orbits, namely  $\{3, 6, 9, 12\}$  and  $\{5, 10\}$ . However each orbit is associated with a unique value of  $i$ , namely  $i = \bar{\alpha}_p(j)$ , the sum of the digits of the  $p$ -adic expansion of any  $j \in \gamma$ .

Theorem 1.12 follows from Theorem 1.13 by:

*Proof of Theorem 1.12.* All we need to prove here, after we see how our functors are constructed, is that there are no non-trivial maps  $A_\gamma \rightarrow B_\beta$  if  $\gamma \neq \beta$ . Such a map,  $f$ , must commute with  $e_\beta$  which is the identity on  $B_\beta$  but is the trivial map on  $A_\gamma$  and so it must be the trivial map.  $\square$

*Proof of Theorem 1.13.* Our concern for the rest of this section is to construct the idempotents  $e_\gamma$  and prove Theorem 1.13. A Hopf algebra,  $A$ , in  $EC(n)$  is equipped with the usual Frobenius and Verschiebung endomorphisms  $F$  and  $V$  (see 4.6). The relation,  $VF = FV = p$ , is a simple calculation. Do not confuse this  $p$  with multiplication by  $p$ . This is  $p$  times the identity in the abelian group of endomorphisms of  $A$ . Thus the ring of endomorphisms of  $A$  which ignores the grading is a module over  $\mathbf{Z}_{(p)}$ . Because the primitive filtration is exhaustive every element of  $A$  is annihilated by some power of  $V$  and hence by some power of  $p$ . This means that the endomorphism ring is also a module over the  $p$ -adic integers  $\mathbf{Z}_p$ .

Given an element  $a \in \mathbf{F}_p$  we can define an endomorphism  $[a]$  of  $A$  to be multiplication by  $a^i$  in dimension  $2i$ . If we replace  $A$  by  $A \otimes \mathbf{F}_{p^n}$ , we have an endomorphism  $[a]$  for each  $a \in \mathbf{F}_{p^n}$ . Hence the endomorphism ring of  $A \otimes \mathbf{F}_{p^n}$  is a module over  $\mathbf{Z}_p[G]$ , the  $p$ -adic group ring over  $G = \mathbf{Z}/(p^n - 1)$ , which is isomorphic to the multiplicative group of the field  $\mathbf{F}_{p^n}$ .

**Remark 4.8** *Although our applications in this paper only require us to study*

evenly graded Hopf algebras, our study of these Hopf algebras would be incomplete without splitting the category into evenly graded Hopf algebras and exterior Hopf algebras on odd degree generators for odd primes. Classical sign arguments force a Hopf algebra with odd degree primitives to be an exterior algebra. The idempotents necessary for this splitting are just  $([1] \pm [-1])/2$ . To check this we need only evaluate on primitives.  $([1] - [-1])/2$  is the identity on odd primitives and trivial on even primitives. The reverse is true for  $([1] + [-1])/2$ .

We want to use  $\mathbf{Z}_p[G]$  to construct idempotents of  $A$ . We need some basic facts to do this. Let  $M$  be an  $R$  module and let  $H$  act on  $R$  and  $M$ . Then we have a map  $j : R \rightarrow \text{End}(M)$  and  $H$  acts on  $f \in \text{End}(M)$  by  $(hf)(x) = h(f(h^{-1}(x)))$ . Then

**Lemma 4.9** *If the map  $j$  is  $H$ -equivariant then the fixed points,  $R^H$  act on the fixed points  $M^H$ .*

*Proof.* If  $m \in M^H$  and  $r \in R^H$  then

$$(jr)(m) = (jh(r))(m) = [h(jr)](m) = h(jr(h^{-1}m)) = h(jr(m)).$$

□

We have our group  $H = \mathbf{Z}/(n)$ .  $H$  is isomorphic to the Galois group of  $\mathbf{F}_{p^n}$  over  $\mathbf{F}_p$ , and its action on the ring  $\mathbf{Z}_p[G]$  corresponds to the action of the Galois group on the units of  $\mathbf{F}_{p^n}$ . We will write  $G$  multiplicatively from now on. We want to show that if an element  $x \in \mathbf{Z}_p[G]$  is fixed by this action then the corresponding endomorphism leaves  $A = (A \otimes \mathbf{F}_{p^n})^H \subset A \otimes \mathbf{F}_{p^n}$  invariant. Hence the endomorphism ring of  $A$  is a module over the fixed point subring  $\mathbf{Z}_p[G]^H$ . By our lemma, we must just show that the map  $\mathbf{Z}_p[G] \rightarrow \text{End}(A \otimes \mathbf{F}_{p^n})$  is  $H$ -equivariant. It is enough to show this on  $[g]$ . Let  $a \in A$  and  $z \in \mathbf{F}_{p^n}$ . Then:

$$j(h([g]))(a \otimes z) = j([g^{p^h}])(a \otimes z)$$

$$\begin{aligned}
&= a \otimes g^{p^h|a|} z \\
&= h(a \otimes g^{|a|} z^{1/p^h}) \\
&= h(j[g])(a \otimes z^{1/p^h}) \\
&= h(j[g])h^{-1}(a \otimes z).
\end{aligned}$$

We construct some idempotents in an even bigger ring,  $W(\mathbf{F}_{p^n})[G]$ , where  $W(\mathbf{F}_{p^n})$  denotes the Witt ring of  $\mathbf{F}_{p^n}$ , i.e., the extension of  $\mathbf{Z}_p$  obtained by adjoining a primitive  $(p^n - 1)^{\text{th}}$  root of unity  $\omega$ . See [Haz78, page 132, Remark 17.4.18]. We will use the notation  $\omega$  for both the root of unity in  $W(\mathbf{F}_{p^n})$  and its mod  $p$  reduction in  $\mathbf{F}_{p^n}$ . For each  $j \in G$  let

$$e_j = \sum_{1 \leq i \leq p^n - 1} \frac{\omega^{-ij} [\omega^i]}{p^n - 1}. \quad (4.10)$$

Note that  $e_j$  is independent of the choice of the primitive root  $\omega$ . One sees easily that the  $e_j$  are orthogonal idempotents, i.e.,

$$\begin{aligned}
e_j e_{j'} &= \begin{cases} e_j & \text{if } j = j' \\ 0 & \text{otherwise} \end{cases} \\
\text{and } \sum_j e_j &= 1.
\end{aligned} \quad (4.11)$$

Now let

$$c_m = \sum_k \omega^{mp^k} \in \mathbf{Z}_p \subset W(\mathbf{F}_{p^n}) \quad (4.12)$$

where the sum is over all distinct roots of the indicated form. Thus the number of summands is some divisor of  $n$  depending on  $m$ . This element is in  $\mathbf{Z}_p$  because it is invariant under the action of the Galois group  $H$ , whose generator sends each root of unity to its  $p^{\text{th}}$  power.

Now let  $\gamma$  be the  $H$ -orbit containing  $j$  and consider the element

$$e_\gamma = \sum_k e_{jp^k}$$

$$\begin{aligned}
&= \sum_{i,k} \frac{\omega^{-ijp^k} [\omega^i]}{p^n - 1} \\
&= \sum_i \frac{c_{-ij} [\omega^i]}{p^n - 1},
\end{aligned}$$

where the first sum is as in (4.12). This element is idempotent by (4.11). It is in  $\mathbf{Z}_p[G] \subset W(\mathbf{F}_{p^n})[G]$  since the coefficients  $c_{-ij}$  all lie in  $\mathbf{Z}_p$ .

Furthermore  $e_\gamma$  is fixed by the Galois action since  $c_{-ijp^k} = c_{-ij}$ , so it lies in the invariant subring  $\mathbf{Z}_p[G]^H$ . This is the idempotent giving the splitting of Theorem 1.11.

In order to verify this, it suffices to show that it behaves appropriately on primitive elements. Let  $x$  be a primitive in dimension  $m$ . Recall that the endomorphism ring of a Hopf algebra acts additively on its primitives, i.e., for endomorphisms  $\alpha$  and  $\beta$  and coefficients  $a, b \in \mathbf{Z}_p$ , we have

$$(a\alpha + b\beta)(x) = a\alpha(x) + b\beta(x)$$

where the coefficients on the right are reduced mod  $p$ . With this in mind we have

$$\begin{aligned}
e_\gamma(x) &= \sum_i \frac{c_{-ij} [\omega^i](x)}{p^n - 1} \\
&= \sum_i \frac{c_{-ij} \omega^{im} x}{p^n - 1} \\
&= \sum_{i,k} \frac{\omega^{im-ijp^k} x}{p^n - 1} \\
&= \sum_k \left( \sum_i \frac{\omega^{i(m-jp^k)}}{p^n - 1} \right) x.
\end{aligned}$$

The inner sum is 1 when  $m = jp^k$  and 0 otherwise, so we have

$$e_\gamma(x) = \begin{cases} x & \text{if } m \in \gamma \\ 0 & \text{otherwise} \end{cases}$$

as desired.  $\square$

In the introduction we defined a function  $\bar{\alpha}_p : \mathbf{Z}/(p^n - 1) \rightarrow \mathbf{Z}$  by taking the sum of the coefficients of the  $p$ -adic expansion. We observed that  $\bar{\alpha}_p$  is constant on orbits  $\gamma$ . From Theorem 1.13, we have:

**Corollary 4.13** *Let  $EC(n)_q = \prod_{\bar{\alpha}_p(\gamma)=q} EC(n)_\gamma$  for all  $0 \leq q \leq n(p-1)$ . Then:*

$$EC(n) = \prod_{0 \leq q \leq n(p-1)} EC(n)_q.$$

This is now what we need for our Hopf algebras. We actually need very little knowledge about  $K(n)_*(K(\mathbf{Z}/(p^i), q))$ . We remind the reader that

$$\overline{K(n)}_*(X) \equiv K(n)_*(X) \otimes_{K(n)_*} \mathbf{F}_p$$

where the  $K(n)_*$  module structure of  $\mathbf{F}_p$  is given by sending  $v_n$  to 1. The following is a restatement of Theorem 1.14.

**Theorem 4.14** ([RW80])  *$\overline{K(n)}_*(K(\mathbf{Z}/(p^i), q))$  is in  $EC(n)_q$  for  $0 < q \leq n$ , and is  $\overline{K(n)}_* \simeq \mathbf{F}_p$  for  $q > n$ .  $\overline{K(n)}_*(K(\mathbf{Z}_{(p)}, q+1))$  is the direct limit of  $\overline{K(n)}_*(K(\mathbf{Z}/(p^i), q))$  and is also in  $EC(n)_q$ .*

*Proof.* The theorem is read off from [RW80, Theorem 11.1] where we see the primitives are all in degrees  $2(p^{i_1} + p^{i_2} + \dots + p^{i_q})$ ,  $0 = i_1 < i_2 < \dots < i_q < n$ , for  $\mathbf{Z}/(p^i)$  and [RW80, Corollary 12.2] for  $\mathbf{Z}_{(p)}$ . In fact, all primitives come from  $\overline{K(n)}_*(K(\mathbf{Z}/(p), q))$ .  $\square$

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