

Commutative Morava homology Hopf algebras

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ABSTRACT. We give the Dieudonné module theory for $\mathbf{Z}/2(p^n - 1)$ -graded bicommutative Hopf algebras over \mathbf{F}_p . These objects arise as the Morava K -theory of homotopy associative, homotopy commutative H -spaces.

1. Introduction

In [DG70, V], Demazure and Gabriel classify commutative unipotent algebraic groups over a perfect field of characteristic p in terms of Dieudonné-modules. Making appropriate translations of terminology, this classifies a category of Hopf algebras in terms of Dieudonné-modules.

Bousfield, in an appendix to [Bou], proves many of the results of interest in an accessible way, adapting and clarifying [DG70, V] for his specific purposes. In particular, he gives much of the structure for $\mathbf{Z}/(2)$ -graded bicommutative Hopf algebras over \mathbf{F}_p . He needs these results for his work with mod p K -theory. We are interested in similar results for Morava K -theory. The Morava K -theory of homotopy commutative H -spaces gives rise to bicommutative $\mathbf{Z}/2(p^n - 1)$ -graded Hopf algebras satisfying certain conditions. Having a Dieudonné module theory for these Hopf algebras gives us a great deal of control over their structure. When the grading is forgotten then Bousfield's results can be applied, so our contribution is really to unravel the gradings in our case and make appropriate definitions so that Bousfield's proofs go through with little or no modification.

Specifically, we want to study the category $\mathcal{C}(n)$ of *commutative Morava Hopf algebras*. These are bicommutative, biassociative, Hopf algebras over \mathbf{F}_p (in our proofs one could substitute any perfect field of characteristic p) which are graded over $\mathbf{Z}/2(p^n - 1)$ ($n > 0$) and have an exhaustive primitive filtration. This last condition, *an exhaustive primitive filtration*, means that some iterate of the reduced coproduct is 0. (Bousfield calls this property “irreducible.”) Bousfield's work analyzes these Hopf algebras quite thoroughly by reducing the $\mathbf{Z}/(2)$ -graded case to the ungraded case by splitting off the sub-algebra generated by odd-dimensional primitives. The study of these Hopf algebras in [HRW97] concentrates on important consequences of the grading. Our goal here is to put the grading into Bousfield's results.

1991 *Mathematics Subject Classification*. Primary 16W30, 55N20.

The first author was partially supported by the National Science Foundation.

[HRW97] shows $\mathcal{C}(n)$ splits (for odd primes) as the product of two categories; $\mathcal{OC}(n)$, whose objects are Hopf algebras generated by primitives in odd degrees, and hence primitively generated exterior algebras, and $\mathcal{EC}(n)$, whose objects are all concentrated in even degrees. (Note that [Bou] proves the same result for $\mathcal{C}(1)$, and his proof also holds without alteration for $\mathcal{C}(n)$.) So, our real category of study will be $\mathcal{EC}(n)$.

In [Bou96], Bousfield shows $\mathcal{C}(1)$ is an abelian category, but again, his proof works just as well for $\mathcal{C}(n)$. Our contribution is to define a functor to Dieudonné modules, to which we again appeal to [Bou] with small alterations.

We define a *Morava-Dieudonné module* to be a $\mathbf{Z}/(p^n - 1)$ -graded abelian group M with endomorphisms F (which multiplies degrees by p), V (which divides degrees by p) : $M \rightarrow M$ such that:

- (i) $FV = p = VF$;
- (ii) for each $x \in M$, there exists $q \geq 1$ with $V^q(x) = 0$.

We denote the category of Morava-Dieudonné modules by $\mathcal{MD}(n)_*$. Note the definition implies the modules are all p -torsion, i.e. $p^q(x) = (FV)^q(x) = F^qV^q(x) = 0$ where we have used (i) and (ii).

Our main theorem is:

THEOREM 1.1. *There is a functor m_* such that $m_* : \mathcal{EC}(n) \rightarrow \mathcal{MD}(n)_*$ is an equivalence of abelian categories.*

Let $\mathbf{Z}/(n)$ act on $\mathbf{Z}/(p^n - 1)$ by $k * j = jp^k$. The orbits are given by $\gamma = \{j, jp, jp^2, jp^3, \dots, jp^n = j\}$. We can see with no work that $\mathcal{MD}(n)_*$ splits as a product of categories $\mathcal{MD}_\gamma(n)_*$ where all elements are in the degrees contained in γ . This translates into a similar theorem, a major result of [HRW97], about $\mathcal{EC}(n)$.

THEOREM 1.2. *As categories:*

$$\mathcal{EC}(n) \simeq \prod_{\gamma} \mathcal{EC}(n)_{\gamma}$$

where $\mathcal{EC}(n)_{\gamma}$ consists of Hopf algebras in $\mathcal{EC}(n)$ with all primitives in degrees 2γ .

This result was proven in [HRW97] so it could be combined with results of [RW80] to show that there is only the trivial map from $K(n)_*(K(\mathbf{Z}/(p^i), s))$ to $K(n)_*(K(\mathbf{Z}/(p^j), t))$ when $s \neq t$.

The motivation for us is Morava K -theory, $K(n)_*(-)$, which is a generalized homology theory with a Künneth isomorphism. The coefficient ring is $K(n)_* \simeq \mathbf{F}_p[v_n, v_n^{-1}]$ where the degree of v_n is $2(p^n - 1)$. By setting $v_n = 1$ we get a $\mathbf{Z}/2(p^n - 1)$ -graded theory, $\overline{K(n)}_*(-)$. If a space X is connected and the loop space of an H -space then $\overline{K(n)}_*(X)$ is in $\mathcal{C}(n)$ for odd primes (there are some complications with commutativity for $p = 2$).

Some examples from [RW80] are

$$m_*(\overline{K(n)}_*(K(\mathbf{Z}/(p), n))) \simeq \mathbf{Z}/(p)$$

in degree $1 + p + \dots + p^{n-1}$ with $V = 0$ and $F(1) = (-1)^{n-1}$ and

$$m_*(\overline{K(n)}_*(K(\mathbf{Z}/(p), n + 1))) \simeq \mathbf{Q}/\mathbf{Z}/(p)$$

in degree $1 + p + \dots + p^{n-1}$ with $V = (-1)^{n-1}p$ and $F(x) = (-1)^{n-1}x$. In addition, for all Eilenberg-Mac Lane spaces except S^1 , the Morava K -theory lies in $\mathcal{EC}(n)$.

There are a couple of additional theorems worth pointing out. The first is that under very mild conditions, see [Bou, Appendix B, Theorem B.4], Sweedler proves what we would call a dual Borel theorem (about the coalgebra structure) in [Swe67a] and [Swe67b]. The second is a comment on Theorem 1.2 splitting $\mathcal{EC}(n)$. It is straightforward to modify the Dieudonné theory above to produce each of these categories. All we have to do is restrict our attention to the functors m_t where $t \in \gamma$.

Background. A Dieudonné module theory has long been developed for graded Hopf algebras, [Sch70]. The authors had been talking about this project for some time when the second author was decimated by the chairmanship of his department. By the time he had recovered, the two papers, [Bou] and [Bou96], were available and the project was easy to complete by co-opting the proofs from Bousfield, [Bou]. Pete Bousfield politely but firmly declined the authors' invitation to be a coauthor. The authors regret his decision and make no claim to originality. We feel the theorems and proofs should be made available however. Following a suggestion of Igor Kriz's we were developing the general Dieudonné module theory for Hopf rings only to find that Paul Goerss had already done it.

2. The Witt algebra, Frobenius and Verschiebung

We review the Witt algebra before constructing the $\mathbf{Z}/2(p^n - 1)$ graded versions we will use.

Let $W = \mathbf{Z}[x_0, x_1, \dots]$. We identify

$$\mathrm{Hom}_{\mathbf{Z}\text{-alg}}(W, A) \cong A^{\mathbf{N}}$$

where \mathbf{N} is the set of non-negative integers, and an algebra homomorphism $f : W \rightarrow A$ is identified with the vector $(f(x_0), f(x_1), \dots)$. Then we define a map

$$w : A^{\mathbf{N}} \rightarrow A^{\mathbf{N}}$$

by

$$(a_0, a_1, \dots) \mapsto (a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2a_2, \dots).$$

We also write w_n for the n th coordinate of w , and hence

$$w_n(a_0, a_1, \dots) = a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^{n-1}a_{n-1}^p + p^n a_n.$$

Note that the map w is injective if A is p -torsion free.

THEOREM 2.1 (Witt). *The image of w is a subgroup of $A^{\mathbf{N}}$.*

This is proved in [Wit36, Satz 1], but a more modern treatment following Lazard can be found on page 40 ff. of [Ser67]. The issue in the proof is: given (a_0, a_1, \dots) and (b_0, b_1, \dots) one needs to find (c_0, c_1, \dots) so that $w(\bar{c}) = w(\bar{a}) + w(\bar{b})$. This can clearly be done over $p^{-1}A$, so the problem is an integrality one.

COROLLARY 2.2. *W has a commutative, associative coproduct determined by declaring $w_n(\bar{x})$ to be primitive for all n . This makes W a bicommutative Hopf-algebra.*

PROOF. The proof is by Yoneda's lemma. We let A range over p -torsion-free \mathbf{Z} -algebras and give $\mathrm{Hom}_{\mathbf{Z}\text{-alg}}(W, A)$ the group structure derived from Theorem 2.1. Precisely, given $f, g \in \mathrm{Hom}_{\mathbf{Z}\text{-alg}}(W, A)$, f is determined by the vector $f(\bar{x}) = \bar{a}$,

g is determined by the vector $g(\bar{x}) = \bar{b}$. Then $f + g$ is determined by the vector \bar{c} where $w(\bar{c}) = w(\bar{a}) + w(\bar{b})$, so $(f + g)(x_i) = c_i$.

This gives a product

$$(2.3) \quad \begin{aligned} \text{Hom}_{\mathbf{Z}\text{-alg}}(W \otimes_{\mathbf{Z}} W, A) &\simeq \\ \text{Hom}_{\mathbf{Z}\text{-alg}}(W, A) \times \text{Hom}_{\mathbf{Z}\text{-alg}}(W, A) &\rightarrow \text{Hom}_{\mathbf{Z}\text{-alg}}(W, A). \end{aligned}$$

By Yoneda's lemma, we get a map of \mathbf{Z} -algebras

$$\Psi : W \rightarrow W \otimes_{\mathbf{Z}} W.$$

Since $\text{Hom}_{\mathbf{Z}\text{-alg}}(W, A)$ is an abelian group, this makes W a cocommutative Hopf-algebra.

To see the $w_n(\bar{x})$ are primitive, we take $A = W \otimes_{\mathbf{Z}} W$ in (2.3) and want that

$$(i_L + i_R)(w_n(\bar{x})) = w_n(\bar{x}) \otimes 1 + 1 \otimes w_n(\bar{x})$$

where $i_L : W \rightarrow W \otimes_{\mathbf{Z}} W$ takes a to $a \otimes 1$, and i_R is defined similarly. So if we write x_i^L for $x_i \otimes 1$ and $x_i^R = 1 \otimes x_i$, i_L is represented by (x_0^L, x_1^L, \dots) and i_R is represented by (x_0^R, x_1^R, \dots) . Hence $(i_L + i_R)$ is represented by polynomials (c_0, c_1, \dots) where $w_n(\bar{c}) = w_n(x^L) + w_n(x^R)$. In other words,

$$(i_R + i_L)(w_n(\bar{x})) = w_n(\bar{c}) = w_n(x^L) + w_n(x^R) = w_n(\bar{x}) \otimes 1 + 1 \otimes w_n(\bar{x}),$$

as was to be shown.

To note that $w_n(\bar{x})$ primitive determines the co-algebra structure, observe that the $w_n(x)$ are polynomial generators of $W \otimes_{\mathbf{Z}} \mathbf{Q}$, so since W is torsion free, there is at most one Hopf algebra structure with the $w_n(x)$ primitive. \square

REMARK 2.4. If we grade W by setting $|x_0| = 2t$ and $|x_i| = p^i 2t$, then the $w_n(\bar{x})$ are homogeneous, so we get a graded Hopf-algebra. We call the resulting Hopf algebra W_t . Another reference for the above is [Kra72].

DEFINITION 2.5. We give a Hopf algebra graded over $\mathbf{Z}/2(p^n - 1)$ by defining

$$W(s, t) \equiv \mathbf{F}_p[x_0, x_1, \dots, x_s] \subset W_t \otimes \mathbf{F}_p.$$

Frobenius and Verschiebung. Bicommutative Hopf algebras over fields of characteristic p come with two self maps, the *Frobenius* F and the *Verschiebung*, V . The Frobenius is the p th power map, i.e. $F(x) = x^p$, which is a map of Hopf algebras since we are in characteristic p , but not of graded Hopf algebras. The Verschiebung can be defined as the dual of the Frobenius map on the dual Hopf algebra, as in [HRW97], or in terms of the coproduct, as in [Bou]. One construction of the Verschiebung is as follows: choose a basis for the bicommutative Hopf algebra of characteristic p , A . Write $\Psi^{p-1}(x)$ as a sum of tensor products of basis elements, $\sum_{i=1}^k a_{i1} \otimes \dots \otimes a_{ip}$. Then

$$V(x) = \sum_{\{i|a_{i1}=\dots=a_{ip}\}} a_{i1}.$$

It is an easy exercise to check that this definition does not depend on the basis chosen, and another exercise to check that it gives a map of Hopf algebras. It doesn't give a map of graded Hopf algebras since $|V(x)| = |x|/p$.

When x is primitive, $V(x) = 0$, so V is 0 on primitively generated bicommutative Hopf algebras. Clearly $FV = VF$ since V is a map of algebras. Also, a direct consequence of the addition as defined above in $\text{Hom}_{\mathbf{Z}\text{-alg}}(A, B)$ (where A is some

bicommutative Hopf algebra over \mathbf{F}_p and B runs through \mathbf{F}_p -algebras) is that FV on A induces multiplication by p in the abelian group $\text{Hom}_{\mathbf{Z}\text{-alg}}(A, B)$. It follows that if A is primitively generated, $\text{Hom}_{\mathbf{Z}\text{-alg}}(A, B)$ is an \mathbf{F}_p -vector space.

We would like to calculate the value of V on $W \otimes \mathbf{F}_p$. We begin with the observation that since F is injective, we can deduce V from FV . Since FV induces multiplication by p in $\text{Hom}_{\mathbf{F}_p\text{-algebra}}(W \otimes \mathbf{F}_p, A)$, we'll begin by analyzing the self-map of W that induces multiplication by p in $\text{Hom}_{\mathbf{Z}\text{-alg}}(W, A)$. Recall that $f : W \rightarrow A$ is represented by a vector (a_0, a_1, \dots) where $a_i = f(x_i)$ and that the p -fold sum of f with itself, which we'll write $[p](f)$ is represented by a vector (c_0, c_1, \dots) where $c_i = ([p]f)(x_i)$ and $pw_n(\bar{a}) = w_n(\bar{c})$. Now observe that

$$(2.6) \quad pw(\bar{a}) = (pa_0, pa_0^p + p^2a_1, \dots, pa_0^{p^n} + p^2a_1^{p^{n-1}} + \dots + p^na_{n-1}^p + p^{n+1}a_n, \dots).$$

So $c_0 = pa_0, c_1 = a_0^p + p(a_1 - p^{p-2}a_0^p)$. If one assumes inductively that $c_j = a_{j-1}^p + pb_j$, for $j \leq i$ one sees that

$$\begin{aligned} w_{i+1}(\bar{c}) &= c_0^{p^{i+1}} + pc_1^{p^i} + \dots + p^i c_i^p + p^{i+1} c_{i+1} \\ &= (pb_0)^{p^{i+1}} + p(a_0^p + pb_1)^{p^i} + \dots + p^i (a_{i-1}^p + pb_i^p) + p^{i+1} c_{i+1} \\ &\equiv pa_0^{p^{i+1}} + \dots + p^i a_{i-1}^{p^2} + p^{i+1} c_{i+1} \pmod{p^{i+2}}. \end{aligned}$$

and hence by equating with (2.6) that $c_{i+1} = a_i^p + pb_{i+1}$. On reduction mod p , it follows that multiplication by p on $\text{Hom}_{\mathbf{F}_p\text{-algebra}}(W_p, A)$ is induced by the map that sends x_i to x_{i-1}^p . Since $F(x_{i-1}) = x_{i-1}^p$, it follows then that $V(x_i) = x_{i-1}$.

3. The functor to Dieudonné-modules

Although, as we have remarked, neither F nor V give maps of graded Hopf algebras, we have maps F^n and V^n in our cyclicly graded category $\mathcal{EC}(n)$. The V^n induce surjections by:

$$\mathbf{F}_p[x_0, \dots, x_{ns}] = W(ns, t) \longrightarrow W(n(s-1), t) = \mathbf{F}_p[x_0, \dots, x_{n(s-1)}].$$

We use the sequence:

$$W(0, t) \xleftarrow{V^n} W(n, t) \leftarrow \dots \leftarrow W(n(s-1), t) \xleftarrow{V^n} W(ns, t) \quad \dots$$

to define

$$m_t : \mathcal{EC}(n) \longrightarrow \mathcal{MD}(n)_t$$

by

$$m_t(A) = \varinjlim \text{Hom}(W(ns, t), A).$$

REMARK 3.1. To get the equivalence of $\mathcal{EC}(n)_\gamma$ and $\mathcal{MD}(n)_\gamma$ we can modify the definition slightly. Instead of using V^n we can replace n with the order of the orbit γ . In particular, when $\gamma = \{0\}$ we are precisely in the ungraded case of [Bou] except with slightly modified, but completely equivalent, definitions for F and V on the image of our functors.

To complete our definition of $m_* : \mathcal{EC}(n) \rightarrow \mathcal{MD}(n)_*$ we need to define the action of F and V on $m_*(A)$. We require a minor departure from the standard definition in order to preserve grading. In our case we use the fact that the image of F on $W(s, t)$ is canonically isomorphic to $W(s, pt)$, i.e. the composite

$$W(s, t) \xrightarrow{\cong} W(s, pt) \hookrightarrow W(s, t)$$

that sends $x_i \mapsto x_i \mapsto x_i^p$ is F . The second map here preserves degree, so is in our category.

Because F and V commute we get a map of sequences

$$\begin{array}{ccccccccccc} W(0, t) & \leftarrow & W(n, t) & \leftarrow & \cdots & \leftarrow & W(n(s-1), t) & \leftarrow & W(ns, t) & \cdots \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\ W(0, pt) & \leftarrow & W(n, pt) & \leftarrow & \cdots & \leftarrow & W(n(s-1), pt) & \leftarrow & W(ns, pt) & \cdots \end{array}$$

which induces the map F

$$m_t(A) = \varinjlim \text{Hom}(W(ns, t), A) \longrightarrow m_{pt}(A) = \varinjlim \text{Hom}(W(ns, pt), A).$$

We now need the map V . The image of V acting on $W(s, t)$ is the sub-Hopf algebra $W(s-1, t)$. There is a canonical map of $W(s, t/p)$ which surjects to this image. In particular, this gives us maps

$$W(ns, t/p) \rightarrow W(ns-1, t) \subset W(ns, t).$$

The corresponding map of sequences

$$\begin{array}{ccccccccccc} W(0, t) & \leftarrow & W(n, t) & \leftarrow & \cdots & \leftarrow & W(n(s-1), t) & \leftarrow & W(ns, t) & \cdots \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\ W(0, t/p) & \leftarrow & W(n, t/p) & \leftarrow & \cdots & \leftarrow & W(n(s-1), t/p) & \leftarrow & W(ns, t/p) & \cdots \end{array}$$

induces V

$$m_t(A) = \varinjlim \text{Hom}(W(ns, t), A) \longrightarrow m_{t/p}(A) = \varinjlim \text{Hom}(W(ns, t/p), A).$$

4. The equivalence of categories

We are now set to prove our main Theorem 1.1.

PROOF. We follow Bousfield's analogous proof in [Bou, Appendix A] very closely. As in his proof and the similar result in [DG70, V, §1.n.4], we verify directly that m_* is an isomorphism of categories rather than constructing an inverse functor from abstract considerations. We remind the reader as we remarked in the introduction that the proof for the $\mathbf{Z}/(2)$ -graded case in [Bou96] implies that $\mathcal{EC}(n)$ is an abelian category. The structure of the rest of the proof is as follows:

- (i) Show m_* is exact (for which we'll need to prove that $W(s, t)$ is projective in the subcategory $\mathcal{EC}(n, s)$ defined below).
- (ii) Show every element of $\mathcal{MD}(n)$ is $m_*(B)$ for some $B \in \mathcal{EC}(n)$.
- (iii) Show m_* is an isomorphism on Hom sets.
- (iv) Show m_* is an isomorphism on the class of objects, which is implied by the previous two points.

We begin by defining $\mathcal{EC}(n, k)$ to be the objects in $\mathcal{EC}(n)$ on which V^{k+1} is 0, and similarly for $\mathcal{MD}(n, k)$.

We observe that $\mathcal{EC}(n, 0)$ is the category of primitively generated objects in $\mathcal{EC}(n)$. This follows from the same fact in the ungraded case, [Swe67a, Lemma 3] (see the corollary immediately after the proof of Lemma 3), because the primitives

of the *ungraded* Hopf algebra are generators, but since the diagonal map on an object of $\mathcal{EC}(n)$ preserves the grading, the primitives of such an object are the direct sum of the homogeneous primitives.

On the other hand $\mathcal{MD}(n, 0)$ can be thought of as restricted \mathbf{F}_p -Lie algebras with trivial bracket ($V = 0$ implies $p = 0$ on objects of $\mathcal{MD}(n, k)$). For $A \in \mathcal{EC}(n, 0)$ we check that $m_*(A) = PA$, the primitives of A . To see this note that $\text{Hom}(W(ns, t), A) = P_t A$ because any map from $W(ns, t)$ to A is 0 on the image of V , so factors through $W(ns, t) \rightarrow W(0, tp^{ns}) = W(0, t)$. Since this factorization is preserved under the maps in the system:

$$\begin{array}{ccc} W(ns, t) & \xrightarrow{V^n} & W(n(s-1), t) \\ \downarrow & & \downarrow \\ W(0, t) & \xrightarrow{=} & W(0, t) \end{array}$$

we get that $m_t(A) = \lim_{\rightarrow} \text{Hom}(W(ns, t), A) = P_t A$. The functor back from $\mathcal{MD}(n, 0)$ is of course the restricted enveloping algebra, which gives an equivalence of categories $\mathcal{EC}(n, 0)$ and $\mathcal{MD}(n, 0)$. For this we apply [MM65, Theorem 6.11] in the ungraded case to see that as an ungraded Hopf-algebra, $A \in \mathcal{EC}(n, 0)$ implies $A = V(PA)$; the restricted enveloping algebra on the primitives. Again since $P(A)$ is just the direct sum of its homogeneous pieces we can consider $P(A)$ as a cyclically graded \mathbf{F}_p -Lie algebra with trivial bracket ($A = V(PA)$ is commutative). Again, since $M = PV(M)$ as an ungraded Lie algebra, the same fact about the diagonal (now on $V(M)$) tells us that this formula holds as a graded Lie algebra.

Next one wants to see that $W(s, t)$ is projective in $\mathcal{EC}(n, s)$. We've established this above when $s = 0$. One can follow the proof of [Bou, Lemma A.12] precisely: it suffices to prove that for each $B \in \mathcal{EC}(n, s)$ that $\text{Ext}_{\mathcal{EC}(n, s)}^1(B, W(s, t)) = 0$. Since B is filtered by the kernels of V^i , with filtration quotients in $\mathcal{EC}(n, 0)$ it suffices to check this when $B \in \mathcal{EC}(n, 0)$. So we look at such an extension

$$(4.7) \quad B \rightarrow \overline{B} \rightarrow W(s, t).$$

Taking both the kernel and cokernel of V of this short exact sequence we can use the snake lemma to get a six term exact sequence relating them. In fact this is two short exact sequences because the map $\overline{B} \rightarrow W(s, t)$ is still onto when restricted to the kernel of V because the kernel of V on $W(s, t)$ is also the image of V^s and the image of V^s is a subset of the kernel of V on B since $B \in \mathcal{EC}(n, s)$. Hence the sequence

$$(4.8) \quad B \rightarrow \overline{B} // V \rightarrow W(s, t) // V = W(0, p^s t)$$

in $\mathcal{EC}(n, 0)$ is exact, and therefore split as $W(0, p^s t)$ is projective in $\mathcal{EC}(n, 0)$. It follows that we can split the inclusion in (4.8) by a map $\overline{B} // V \rightarrow B$, and hence can split the inclusion in (4.7).

Now we need to show that $m_*(-)$ is exact. It is obviously left exact, so the issue is showing that if $A \xrightarrow{g} B$ is onto, and $z \in m_t(B)$, then $z = g_*(y) = g \circ y$ for $y \in m_t(A)$. Choose s so z is given by a map $f : W(ns, t) \rightarrow B$. Choose lifts of $f(x_i)$, $i = 0, \dots, ns$ in A . Choose m so V^m annihilates all those lifts. Let $B_m \subseteq B$ be the kernel of V^m and similarly $A_m \subseteq A$. Then f factors through B_m , and let

f^*A_m be the pullback:

$$\begin{array}{ccc} f^*A_m & \longrightarrow & A_m \\ p \downarrow & & \downarrow \\ W(ns, t) & \longrightarrow & B_m. \end{array}$$

If $n(k+s) \geq m$, then $fV^{nk} : W(n(k+s), t) \rightarrow B_m \rightarrow B$ also represents z , and since p is surjective the projectivity of $W(n(k+s), t)$ allows us to lift V^{nk} to f^*A_m , and hence fV^{nk} to A as we require.

Now we need that $m_*W(s, t)$ is projective in $\mathcal{MD}(n, s)$. We do as we did for $W(s, t)$. It suffices to check that $\text{Ext}_{\mathcal{MD}(n, s)}^1(B, m_*W(s, t)) = 0$ for $B \in \mathcal{MD}(n, 0)$. So if we have an extension of Dieudonné-modules,

$$(4.9) \quad B \rightarrow \overline{B} \rightarrow m_*W(s, t)$$

we can observe by exactness of m_* that the kernel of V is the image of V^s on $m_*W(s, t)$, so that we have an extension

$$(4.10) \quad B \rightarrow B/VB \rightarrow m_*W(s, t)/Vm_*W(s, t)$$

in $\mathcal{MD}(n, 0)$. But (4.10) splits by the equivalence between $\mathcal{MD}(n, 0)$ and $\mathcal{EC}(n, 0)$, and it follows that (4.9) splits.

Now we wish to prove that

$$(4.11) \quad m_* : \text{Hom}_{\mathcal{EC}(n)}(W(s, t), A) \cong \text{Hom}_{\mathcal{MD}(n)}(m_*W(s, t), m_*A).$$

It clearly suffices to show this when $A \in \mathcal{EC}(n, s)$ since the kernels of V^{s+1} on A and m_*A is where morphisms from $W(s, t)$ and $m_*W(s, t)$ land, and the functor m_* commutes with this kernel.

By exactness, it also suffices to assume $A \in \mathcal{EC}(n, 0)$. But then

$$\begin{aligned} \text{Hom}_{\mathcal{EC}(n)}(W(s, t), A) &= \text{Hom}_{\mathcal{EC}(n)}(W(s, t)//V, A) \\ &= \text{Hom}_{\mathcal{EC}(n)}(W(0, p^s t), A) = P_{p^s t}A \end{aligned}$$

and

$$\begin{aligned} \text{Hom}_{\mathcal{MD}(n)}(m_*W(s, t), m_*A) &= \text{Hom}_{\mathcal{MD}(n)}(m_*W(s, t)/V, m_*A) \\ &= \text{Hom}_{\mathcal{MD}(n)}(m_*(W(s, t)//V), m_*A) = \\ &= \text{Hom}_{\mathcal{MD}(n)}(m_*W(0, p^s t), m_*A) = P_{p^s t}A. \end{aligned}$$

Next we need to show that every $M \in \mathcal{MD}(n)$ is m_*B for some B . To do this, we take M and realize it as a cokernel (see below):

$$(4.12) \quad \bigoplus_I m_*W(s_i, t_i) \rightarrow \bigoplus_J m_*W(s_j, t_j) \rightarrow M.$$

Since m_* is a direct limit, it commutes with sums, and we get $M = m_*B$ where B is the cokernel in

$$(4.13) \quad \bigotimes_I W(s_i, t_i) \rightarrow \bigotimes_J W(s_j, t_j) \rightarrow B.$$

So it remains to prove that M can be written as in (4.12). Let $x \in M_t$. Then $V^r x = 0$ for some r . $m_*W(r-1, t)$ is

$$\mathbf{Z}/(p) \oplus \mathbf{Z}/(p^2) \oplus \cdots \oplus \mathbf{Z}/(p^{r-1}) \oplus \mathbf{Z}/(p^r) \oplus \mathbf{Z}/(p^r) \oplus \cdots$$

where if we denote the generator of the i th summand by e_i , then

$$\begin{aligned} V(e_i) &= e_{i-1} \text{ if } i \leq r, V(e_i) = pe_{i-1} \text{ if } i > r \\ F(e_i) &= e_{i+1} \text{ if } i \geq r, F(e_i) = pe_{i+1} \text{ if } i < r, \end{aligned}$$

and the degrees are determined by $|e_r| = t$. Then $m_*W(r-1, t)$ is the free Dieudonné-module on a class e_r in dimension t with $V^r(e_r) = 0$, and there is a map of Dieudonné-modules $m_*W(r-1, t) \rightarrow M$ defined by taking e_r to x . It follows there is a surjective map from a sum of $m_*W(s_j, t_j)$ to M , and that we can realize M as a cokernel as in (4.12). Hence $M = m_*B$ for some B .

Finally, to prove m_* is an equivalence of categories, it suffices to show that $m_* : \text{Hom}_{\mathcal{EC}(n)}(B, A) \rightarrow \text{Hom}_{\mathcal{MD}(n)}(m_*B, m_*A)$ is an isomorphism. We do this by showing any $B \in \mathcal{EC}(n)$ can be realized as a cokernel as in (4.13). Once we know this, we get a map of exact sequences

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{EC}(n)}(B, A) & \longrightarrow & \text{Hom}_{\mathcal{MD}(n)}(m_*B, m_*A) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{EC}(n)}(\otimes_J W(s_j, t_j), A) & \longrightarrow & \text{Hom}_{\mathcal{MD}(n)}(\oplus_J m_*W(s_j, t_j), m_*A) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{EC}(n)}(\otimes_I W(s_i, t_i), A) & \longrightarrow & \text{Hom}_{\mathcal{MD}(n)}(\oplus_I m_*W(s_i, t_i), m_*A). \end{array}$$

The third and fourth horizontal maps are isomorphism since in both categories Hom out of a sum is the product of Hom 's and we've already checked (4.11). So by the five lemma, we get the desired isomorphism. \square

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