

**THE IMPOSSIBLE MADE EASY:
LEARNING TO CALCULATE WITH
GENERALIZED COHOMOLOGY**

W. STEPHEN WILSON

A *topologist* studies topological spaces and continuous maps. The typical example of a nice topological space is a *CW complex*; i.e., a space built up from cells. Let us say we have built X . We can add an n -dimensional cell, D^n , to X using any continuous map

$$f : S^{n-1} \cong \partial D^n \rightarrow X;$$

just glue D^n to X by identifying $x \in S^{n-1}$ with $f(x) \in X$. We get a space $Y = XU_f D^n$. In this way we can build a large class of topological spaces which have a certain amount of geometric intuition behind them.

A *homotopy theorist* feels that there are far too many topological spaces and continuous maps to deal with effectively. We immediately put an equivalence relation on the continuous maps from X to Y . We say $f \sim g$, f is *homotopic to* g , if f can be continuously deformed into g ; i.e., if there is a continuous map $F : X \times I \rightarrow Y$, ($I = [0, 1]$), such that $F | X \times 0 = f$ and $F | X \times 1 = g$. This is an equivalence relation and we denote the set of equivalence classes by $[X, Y]$ and call this the *homotopy classes of maps from X to Y* . We say X and Y are

homotopy equivalent or of the *same homotopy type* if we have maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f \sim I_X$ and $f \circ g \sim I_Y$. Two fundamental problems in homotopy theory are to determine if X and Y are of the same homotopy type and to compute $[X, Y]$.

To attempt to solve those problems we move on now to *algebraic topology*. Here we want to have a rule which assigns some algebraic object to every space; $X \mapsto E_*X$. This may just be a set, or have some complicated algebraic structure: groups, rings, algebras, etc. For every map $f : X \rightarrow Y$ we want a corresponding algebraic map $f_* : E_*X \rightarrow E_*Y$. If $f \sim g$ we want $f_* = g_*$. However, this property usually holds for most algebraic invariants anyway, so we are usually forced to go to homotopy theory if we use these algebraic techniques. (We also use algebraic objects E^*X where the algebraic map reverses direction from that of the topological map.) So if $f_* \neq g_*$ then $f \not\sim g$. If $X \sim Y$, then $E_*X \cong E_*Y$. The more algebraic theories we have, the better the chance of distinguishing two maps. The richer the algebraic structure, the more difficult it is to have an isomorphism. For example, it is much easier for two sets to be the same than for them to be isomorphic as groups or rings. Of course a third thing we want is computability, which we usually do not have. It is much easier to define algebraic invariants than to compute them.

We will stick to a special type of algebraic invariant, *generalized (co)homology theories*. There are a bunch of axioms for standard homology, namely it can be defined for pairs of spaces, is a homotopy functor and has some exactness properties. The final axiom for standard homology gives the homology of a point.

If this is dropped, we have axioms for a generalized homology theory.

A generalized cohomology theory consists of an infinite collection of abelian groups, $G^*(X) = \{G^k(X)\}$ and usually has even more structure, for example we will only consider those which give graded rings, i.e. have a pairing

$$G^i(X) \otimes G^j(X) \rightarrow G^{i+j}(X).$$

A theorem of Ed Brown's says that there is always a space \underline{G}_k such that

$$G^k(X) \simeq [X, \underline{G}_k].$$

Furthermore, the axioms imply that $G^*(X) \cong G^{*+1}(\Sigma X)$, where ΣX is the suspension of X . It is a homotopy theoretic fact that

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

where ΩY is the loop space of Y (i.e. the topological space of all maps of the unit interval into Y which start and stop at the same "base" point). Combined, we get

$$[X, \underline{G}_n] \cong G^n X \cong G^{n+1} \Sigma X \cong [\Sigma X, \underline{G}_{n+1}] \cong [X, \Omega \underline{G}_{n+1}]$$

which can be used to show that $\Omega \underline{G}_{n+1} \cong \underline{G}_n$. From this we have that

$$G^n(X) \simeq \lim[\Sigma^k X, \underline{G}_{n+k}].$$

The limit of \underline{G}_{n+k} is the n -th suspension of G , the stable object representing the generalized cohomology theory. Generalized cohomology theories go hand in hand with generalized homology theories. For cohomology we have stable maps

$$G^n(X) \simeq \{X, \Sigma^n G\}.$$

For homology it is

$$G_n(X) \simeq \{S^n, X \wedge G\}.$$

Our first examples of algebraic invariants of this sort are the usual mod 2 homology, H_*X , and cohomology, H^*X . They both satisfy our homotopy condition. The cohomology is the dual to the homology which is just a collection of $Z/2$ vector spaces $H_*X = \{H_iX\}_{i \geq 0}$. H_nX is defined using maps of generalized n -dimensional triangles into X . In particular, H_nX tells us something about how the n -cells of X are related to the $n+1$ dimensional cells and the $n-1$ dimensional cells. Of course the first thing we do is invent homological algebra to deal with homology and get away from the geometry.

The cohomology has more structure than just a collection of vector spaces. Applying $H^*(-)$ to the diagonal map $\Delta : X \rightarrow X \times X$ we get

$$H^*X \leftarrow H^*(X \times X) \cong H^*X \otimes H^*X .$$

The isomorphism is the Künneth theorem but we must define the tensor product for this to make sense. We have

$$H^n(X \times X) \cong \bigoplus_{i+j=n} H^iX \otimes H^jX .$$

What we have now is an algebra, or more precisely, a *graded algebra*. It allows us to multiply $x_i \in H^iX$ and $x_j \in H^jX$ to get an element $x_i x_j \in H^{i+j}X$. This gives us a richer, stronger structure of the sort we want. It is easy to find X and Y with $H^*X \cong H^*Y$ as collections of vector spaces but not as graded algebras. Dually, the homology also has a more complex structure. The diagonal gives

$$\begin{array}{ccc} H_*X & \longrightarrow & H_*(X \times X) \cong H_*X \otimes H_*X \\ x & \xrightarrow{\psi} & \Sigma x' \otimes x'' \end{array}$$

with the degree of $x = \text{sum of degrees of } x' \text{ and } x''$, we call this structure a *coalgebra*. It contains the same information as the dual algebra in cohomology contains.

Our interest is in calculations. The mod 2 homology of spaces is generally assumed to be something we can calculate. In fact, each space is a special problem which has to be done in its own way; there is no “one way” to calculate the homology of spaces. It still takes a great deal of hard work to calculate these groups, if they can be calculated at all.

There are several things which make it possible to calculate the standard mod 2 homology of a space. The first is that the groups are vector spaces, a structure much easier to get a grip on than arbitrary abelian groups. The second is the Künneth isomorphism discussed above which adds the extra structure. The Künneth isomorphism is useful for a number of reasons. It allows you to calculate the homology of products of spaces you know the homology for immediately, a non-trivial extension of what you can do. Better than that is that many spaces can be constructed out of products of other spaces. A common example is when $X = \Omega Y$. If we know the homology of X we can approach the homology of Y because you can build Y from products of X . You don't get the complete answer for free, but you get a very good starting place for your calculations.

Next, the Künneth isomorphism allows you to put more structure on your homology. It is fairly common for spaces of interest to have a multiplication: $X \times X \rightarrow X$. Because of the Künneth isomorphism this puts a multiplication on the homology (or a comultiplication on the cohomology). Combined with the already existing coalgebra structure, we now have a Hopf algebra.

If the multiplication on X is homotopy commutative (and it frequently is) then our Hopf algebra is bicommutative and such objects form an abelian category which therefore has kernels and cokernels etc. When we do this, it is more than just adding structure to the homology so we can use it to distinguish our space from other spaces, it also helps us compute it. For example, there are structure theorems for Hopf algebras, so if we know our answer is a Hopf algebra, the possibilities are severely restricted. Furthermore, the abelian category aspect gives us many computational tools which are not available without it.

We want more algebraic machinery than just standard homology. The next example we offer is the collection of generalized homology theories called Morava K -theories. For each odd prime, p , and each $n > 0$, there is a theory $K(n)_*(-)$. The coefficient ring, i.e., the Morava K -theory of a point, is $K(n)_* \cong \mathbb{Z}/p[v_n, v_n^{-1}]$, with the degree of v_n equal to $2(p^n - 1)$. Although the coefficient ring is graded, we see that every non-zero element is invertible. This makes it into a “graded field.” Thus, much like our standard mod 2 homology example, our generalized homology here is also a “vector space.” This property also gives us a Künneth theorem,

$$K(n)_*(X \times Y) \cong K(n)_*X \otimes K(n)_*Y ,$$

which also leads us to Hopf algebras. In other words, most of the good properties of standard mod p homology carry over to Morava K -theory and, in practice, they have been found to be fairly computable for much the same reasons standard homology is but with very different results and applications.

The example we are interested in is another matter entirely. It is complex bordism. We want to define a sequence of abelian

groups $\Omega_n(X)$. We use manifolds to do this. Manifolds are a much better understood class of topological spaces than the general X we wish to study. We use this understanding of manifolds to study the general X and in the process find new information about the manifolds themselves.

We begin by considering all maps of all n -dimensional manifolds into X ,

$$M^n \xrightarrow{f} X .$$

There are too many such manifolds and maps. So, much like we did when we went to homotopy theory, or when homology is defined using triangles, we put an equivalence relation on these maps. If we have another map, $g : N^n \rightarrow X$, we say f and g are equivalent, or bordant, if there is an $n + 1$ dimensional manifold W^{n+1} and map $F : W^{n+1} \rightarrow X$, such that the boundary, ∂W^{n+1} , of W^{n+1} is the disjoint union of M^n and N^n ; and F restricted to this boundary is the disjoint sum of f and g . Let the equivalence classes be $\Omega_n X$. It is a finitely generated abelian group. All of the axioms for a generalized homology theory can be verified geometrically, or we could easily build a stable object representing it. Let O_n be the n -th orthogonal group and BO_n its classifying space. Take the Thom space of the universal bundle (the one point compactification of the total space of the bundle) to get MO_n . Our maps

$$\begin{array}{ccc} \xi_{n-1} \oplus R & \longrightarrow & \xi_n \\ \downarrow & & \downarrow \\ BO_{n-1} & \longrightarrow & BO_n \end{array}$$

give rise to

$$\Sigma MO_{n-1} \longrightarrow MO_n$$

and Thom transversality gives an isomorphism

$$\Omega_n X \cong \varinjlim_{i \rightarrow \infty} [S^{n+i}, MO_i \wedge X] .$$

We usually denote $\Omega_n X$ by $MO_n X$. A generalized cohomology theory can also be defined as

$$MO^n X \cong \varinjlim_{i \rightarrow \infty} [\Sigma^{i-n} X, MO_i] .$$

This is called unoriented cobordism. This is not so exciting because it is really just a bunch of copies of the standard mod 2 homology. To get something more useful, all we need to do is put a little structure on the manifolds we use. In particular, we assume that the stable normal bundle has a complex structure, induced by a map

$$M^n \rightarrow BU ,$$

and that this structure restricts from W^{n+1} to the boundary when we define bordism. We now get complex bordism, $MU_n X$. Again, we have Thom spaces and

$$MU_n X = \varinjlim_{i \rightarrow \infty} [S^{2i+n} X, MU_i \wedge X]$$

and

$$MU^n X = \varinjlim_{i \rightarrow \infty} [\Sigma^{2i-n} X, MU_i] .$$

For good reasons, MU occupies a special place in homotopy theory. It lies half way between the sphere, a main object of study in homotopy theory, and standard homology, the main tool in algebraic topology. So, it is a tool which is closer to the goal. In particular, we have maps

$$S^0 \longrightarrow MU \longrightarrow K(\mathbb{Z}/(p))$$

where S^0 is the “stable” sphere and $K(\mathbb{Z}/(p))$ is the stable object representing standard mod p cohomology. The homology

of S^0 is known and nearly trivial but its homotopy groups, although of great interest, are too tough to handle. $K(\mathbb{Z}/(p))$ has nearly trivial homotopy groups and although you can calculate its homology, (it is the dual of the Steenrod algebra), it is pretty nasty. MU , in the middle, has very nice homotopy groups and homology! Both are just (graded) polynomial algebras on generators in every even degree. In principle this seems manageable.

MU is central to the study of homotopy theory and Devinatz, Hopkins and Smith have proven the theorem which every homotopy theorist dreams about: they solved a homotopy theory problem completely in terms of an algebraic invariant. Namely, they showed that a stable self map of X is trivial after iteration if and only if the algebra map induced on $MU_*(X)$ is trivial after iteration.

This is a great theorem and establishes (along with other results) the centrality and importance of MU in homotopy theory. However, if we want to be able to use it for more problems, then the computability must be enhanced. Just a few years ago, there were very few examples of non-trivial calculations of the complex cobordism of interesting spaces.

Today, as a result of a recent breakthrough, the complex cobordism of many standard spaces has now been calculated and the answers are quite nice.

First, an old theorem of Quillen's helps us simplify things quite a bit. If you localize $MU_*(X)$ at a fixed prime p then it falls apart into a lot of copies of a much smaller theory called Brown-Peterson homology, which contains all of the same information.

The coefficient ring is

$$BP_* \simeq \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

where the degree of v_n is $2(p^n - 1)$. These v_n are the same v_n we saw some time ago when we discussed Morava K -theories. There is a *Morava Structure Theorem* which relates the Morava K -theories to the Brown-Peterson cohomology of a space.

It turns out that there is a “nice” answer when you calculate the Brown-Peterson cohomology of a space. M is called *Landweber Flat* if M has the property that v_n is injective on $M/(p, v_1, \dots, v_{n-1})M$ for all n . This property turns out to give Künneth isomorphisms and Hopf algebras where we have no right to expect them. It also turns out to be a property that lots and lots of standard spaces in homotopy theory produce when we take their Brown-Peterson cohomology.

To get this, we start with Morava K -theory, which can usually be calculated if necessary, and work from there. The main theorem of Ravenel, Wilson, and Yagita, [RWY98], is that if the Morava K -theory of a space is concentrated in even degrees, then the Brown-Peterson cohomology is too and is Landweber Flat. The condition of even degree Morava K -theory might seem overly restrictive, but there are many examples where it holds. In particular, it is true for the space BO , all Eilenberg-MacLane spaces, finite Postnikov systems, QS^{2n} , the image of J space, the classifying spaces for the symmetric groups and lots of other groups as well.

Since then, from the work of Kashiwabara and Wilson, we can generalize results from [RWY98]. We assume we have maps of

H -spaces

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$$

with the composition trivial. If the Morava K -theory gives an exact sequence of bicommutative Hopf algebras for all Morava K -theories

$$K_* \longrightarrow K_*X_1 \longrightarrow K_*X_2 \longrightarrow K_*X_3 \longrightarrow$$

and $BP^*(X_2)$ and $BP^*(X_3)$ are Landweber Flat, then $BP^*(X_1)$ is the cokernel of f_2^* and is also Landweber Flat.

If instead, we have

$$\longrightarrow K_*X_1 \longrightarrow K_*X_2 \longrightarrow K_*X_3 \longrightarrow K_*$$

for all Morava K -theories and $BP^*(X_1)$ and $BP^*(X_2)$ are Landweber Flat, then $BP^*(X_3)$ is the kernel of f_1^* and is also Landweber Flat.

These techniques allow for the computation of the Brown-Peterson cohomology of a large number of standard spaces in homotopy theory.

REFERENCES

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DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN

E-mail address: wsw@math.jhu.edu